Diploma Programme

Mathematics: applications and interpretation teacher support material

First assessment 2021

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IB mission statement

The International Baccalaureate aims to develop inquiring, knowledgeable and caring young people who help to create a better and more peaceful world through intercultural understanding and respect.

To this end the organization works with schools, governments and international organizations to develop challenging programmes of international education and rigorous assessment.

These programmes encourage students across the world to become active, compassionate and lifelong learners who understand that other people, with their differences, can also be right.

IB learner profile

The aim of all IB programmes is to develop internationally minded people who, recognizing their common humanity and shared guardianship of the planet, help to create a better and more peaceful world.

As IB learners we strive to be:

INQUIRERS
We nurture our curiosity, developing skills for inquiry and research. We know how to learn independently and with others. We learn with enthusiasm and sustain our love of learning throughout life.

KNOWLEDGEABLE
We develop and use conceptual understanding, exploring knowledge across a range of disciplines. We engage with issues and ideas that have local and global significance.

THINKERS
We use critical and creative thinking skills to analyse and take responsible action on complex problems. We exercise initiative in making reasoned, ethical decisions.

COMMUNICATORS
We express ourselves confidently and creatively in more than one language and in many ways. We collaborate effectively, listening carefully to the perspectives of other individuals and groups.

PRINCIPLED
We act with integrity and honesty, with a strong sense of fairness and justice, and with respect for the dignity and rights of people everywhere. We take responsibility for our actions and their consequences.

OPEN-MINDED
We critically appreciate our own cultures and personal histories, as well as the values and traditions of others. We seek and evaluate a range of points of view, and we are willing to grow from the experience.

CARING
We show empathy, compassion and respect. We have a commitment to service, and we act to make a positive difference in the lives of others and in the world around us.

RISK-TAKERS
We approach uncertainty with forethought and determination; we work independently and cooperatively to explore new ideas and innovative strategies. We are resourceful and resilient in the face of challenges and change.

BALANCED
We understand the importance of balancing different aspects of our lives—intellectual, physical, and emotional—to achieve well-being for ourselves and others. We recognize our interdependence with other people and with the world in which we live.

REFLECTIVE
We thoughtfully consider the world and our own ideas and experience. We work to understand our strengths and weaknesses in order to support our learning and personal development.

The IB learner profile represents 10 attributes valued by IB World Schools. We believe these attributes, and others like them, can help individuals and groups become responsible members of local, national and global communities.

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Aims and overview

Welcome to the mathematics teacher support material (TSM). This TSM is designed to assist both new and experienced teachers to build or revise their course design so that it reflects the aims and objectives of the mathematics courses.

The TSM is designed to:

- support experienced and inexperienced teachers alike in structuring and delivering a course
- support teachers with the organization of practical and investigative work
- complement IB professional development.

The TSM is structured to cover generic issues such as the approaches to teaching and learning and TOK and how these relate to mathematics, as well as subject-specific considerations for the teaching of mathematics.

There are three sections to the TSM, which are organized as follows.

- Structuring the courses and making connections—practical advice and suggestions on the organization of classes and an overview of the structure of the courses.
- The "toolkit"—exemplar classroom activities to support the development of inquiry, proof, modelling, and the use of technology. These activities can be used as they stand or can be adapted. They are designed to give teachers guidance and to encourage teachers to develop their own resources. They also contain downloadable materials that can be used with students.
- Assessment—practical advice from experienced teachers and examiners on preparing students for the IA and the HL paper 3.

This TSM has been written by experienced practitioners to support teachers in designing and delivering this subject in a variety of different schools. It is not intended to be prescriptive or an exhaustive way of addressing every issue.

Mathematics: applications and interpretation at a glance

This course recognizes the increasing role that mathematics and technology play in a diverse range of fields in a data-rich world. As such, it emphasizes the meaning of mathematics in context by focusing on topics that are often used as applications or in mathematical modelling. To give this understanding a firm base, this course also includes topics that are traditionally part of a pre-university mathematics course such as calculus and statistics.

The course makes extensive use of technology to allow students to explore and construct mathematical models. Mathematics: applications and interpretation will develop mathematical thinking, often in the context of a practical problem and using technology to justify conjectures.

Students who choose this subject at Standard (SL) or Higher Level (HL) should enjoy seeing mathematics used in real-world contexts and to solve real-world problems. Students who wish to take Mathematics: applications and interpretation at HL will have good algebraic skills and experience of solving real-world problems. They will be students who get pleasure and satisfaction when exploring challenging problems and who are comfortable undertaking this exploration using technology.

Approaches to teaching and learning

Approaches to teaching and learning across the Diploma Programme refer to deliberate strategies, skills and attitudes that permeate the teaching and learning environment. These approaches and tools, intrinsically linked with the IB learner profile attributes, enhance student learning and assist student preparation for the Diploma Programme assessment and beyond.

The aims of approaches to teaching and learning in the Diploma Programme are to:

- empower teachers as teachers of learners as well as teachers of content
- empower teachers to create clearer strategies for facilitating learning experiences in which students are more meaningfully engaged in structured inquiry and greater critical and creative thinking
- promote both the aims of individual subjects (making them more than course aspirations) and linking previously-isolated knowledge (concurrency of learning)
- encourage students to develop an explicit variety of skills that will equip them to continue to be actively engaged in learning after they leave school, and to help them not only obtain university admission through better grades but also prepare them for success during tertiary education and beyond
- enhance further the coherence and relevance of the students’ Diploma Programme experience
- allow schools to identify the distinctive nature of an IB Diploma Programme education, with its blend of idealism and practicality.

The five approaches to learning (developing thinking skills, social skills, communication skills, self-management skills and research skills) along with the six approaches to teaching (teaching that is inquiry-based, conceptually-focused, contextualized, collaborative, differentiated and informed by assessment) encompass the key values and principles that underpin IB pedagogy.

More advice and support on these approaches to teaching and learning can be found in the "Approaches to teaching and approaches to learning" section in the subject guide. Additionally, a suite of materials on approaches to teaching and learning in the Diploma Programme is available on the programme resource centre. The guidance given below builds on these resources.

Approaches to teaching and learning and mathematics

The following articulation of the approaches to teaching and learning is for guidance only and to demonstrate how mathematics as a course facilitates the development of these skills. The links and examples given are not exhaustive, and teachers and students may identify other ways in which these skills are linked to their teaching and learning within mathematics.

Six approaches to teaching

1. Teaching based on inquiry
2. Teaching focused on conceptual understanding
3. Teaching developed in local and global contexts
4. Teaching focused on effective teamwork and collaboration
5. Teaching differentiated to meet the needs of all learners
6. Teaching informed by assessment (formative and summative)

Five approaches to learning

1. Thinking skills
2. Social skills
3. Communication skills
4. Self-management skills
5. Research skills

Approaches to teaching

Inquiry and mathematics

The aims of the mathematics courses emphasize developing students' curiosity and enabling them to use external resources so that they can independently extend their understanding of mathematics. These aims can be achieved through mathematical inquiry. Inquiry based teaching in DP mathematics is a pedagogical approach which allows students to develop conceptual understanding.

Teaching based on inquiry

The idea behind inquiry-based teaching in IB programmes is to develop students' natural curiosity together with the skills of self-management, thinking, research and collaborative learning so that they can become motivated and autonomous lifelong learners.

There are different types of inquiry-based learning, and these include:

- experiential learning
- problem and project-based learning
- discovery learning.

The most significant aspect of inquiry-based teaching is that students are actively engaged in their own learning, constructing their own understandings.

DP mathematics teachers should provide students with opportunities to learn through mathematical inquiry. Lesson plans should accommodate appropriate levels of inquiry (structured, guided, open-ended) that suits different students' needs. In a classroom where inquiry-based teaching is happening, there is much interaction between students, and between students and teacher. The teacher's primary role in such a setting is to promote questions and to facilitate the learning process.

Guiding or essential mathematical questions in the form of facts, concepts and debatable knowledge encourage the learner's curiosity. Students have a degree of freedom to make decisions about how to proceed in their learning, which most often progresses from the concrete towards the abstract.

Conceptual understanding and mathematics

In DP mathematics courses, conceptual understandings are key to promoting deep learning. These conceptual understandings are supported by twelve fundamental concepts which relate with varying emphasis to each of the five topics. Teachers can use these concepts to develop the curriculum. Schools may identify and develop additional concepts to meet local circumstances and curriculum requirements.

Each topic in the guide begins with a description of the essential understandings of the topic, suggestions of concepts fundamental to the topic, and statements about conceptual understanding relevant to the content within the topic.

Teaching focused on conceptual understanding

An important motivation for conceptually focused teaching in IB programmes is to help students build their ability to engage with significant and complex ideas. Equally valuable are the discussions of the "essential understandings" behind a topic, which can help students get to the heart of **why** they are learning what they are learning.

To appreciate the role of concepts in building lasting and significant understandings, it is helpful to think of concepts as the building blocks of students' cognitive frameworks. When they are learning at a conceptual level, students are integrating new knowledge into their existing understandings. They learn how seemingly discrete topics are connected and are ready to transfer their learning to new contexts. A subject emerges for them in a holistic light. In a classroom where conceptually focused teaching is happening, there is continuous movement between facts and what they mean, with students asking why the facts matter as a natural part of their learning process.

Local and global contexts and mathematics

The structure of the mathematics guide provides links to real-life applications, where appropriate, that allow students to contextualize their mathematical learning. The topics in the course have many applications to other disciplines. Mathematical concepts should be taught in a real-world context where appropriate to help students understand local and global phenomena. Students should be given the right content and context in order for them to be able to interpret the given mathematical concept in the global sense. For example, students can learn exponential growth and decay within the context of spread of disease in order to better understand the spread of cholera in Africa.

Teaching developed in local and global contexts

As young individuals and as members of local and global communities, students make sense of the world through their life experiences and the world around them. IB programmes emphasize contextualized teaching because the more students can relate their learning to real-life contexts, the more likely they are to engage with it. IB programmes also enable students to apply their learning; contextualized teaching, like conceptually-focused teaching, helps students to get to the heart of why they are learning what they are learning.

In order to appreciate the role of contexts for relevant learning, it is helpful to think of contexts as students' frames of reference. When they are learning in a contextualized way, students are grounding abstract ideas and new information in familiar real-life situations. In a classroom in which contextualized teaching is happening, concepts and theories are related to accessible and meaningful examples, illustrations and stories, which in turn inform further conceptual and theoretical understandings.

Effective teamwork and collaboration and mathematics

The mathematics courses encourage students to develop an awareness of different approaches and different interpretations of mathematics. Being able to discuss and share approaches to the mathematics content being studied and justify different interpretations provides a rich and meaningful learning opportunity. Students can develop their ability to listen to and respond to one another respectfully and critically, and at times may find that their own approach or interpretation can be influenced by the views of others and that they can influence the approach or interpretation of others.

In mathematics, activities can be developed that encourage students to work effectively in teams. This may involve collaboration at the start of an activity to gather ideas, information or data; during the activity where students take on different roles, for instance analysing data, checking the work of another student, giving feedback on an approach or critiquing an interpretation of a model which helps the group make further progress; and at the end of an activity where, for instance, they might present their approaches or interpretations to the class as a group.

Group work could include using software to create activities in the form of a jigsaw or dominoes set together, a student creating a problem to be solved by one or more classmates, or mind mapping together at the start of a small project. Document-sharing packages can be used in and out of the mathematics classroom as an efficient way for students to share and create their study notes per topic. At the end of a unit of work, students can work in groups to pool their understandings and their understanding of the connections between a topic and others they have studied. Teachers can ask students to recap in groups at the end of each unit by adding their ideas to a poster or to an online padlet.

The exploration internal assessment provides a very valuable learning experience for students to peer review each other's work by providing constructive feedback on their topic and their interpretation and understanding of the assessment criteria.

Teaching focused on effective teamwork and collaboration

IB programmes acknowledge that learning is a social activity. Students and teachers come together, each with unique life experiences, beliefs, ideas, strengths and weaknesses. Learning is the result of these complex interactions between unique individuals.

An important aspect of the learning process is regular feedback from students to teachers on what they have and have not yet understood. Concrete and constructive feedback from teachers to students on performance is similarly crucial for learning to take place.

Meeting the needs of all learners and mathematics

The structure of the course allows teachers to choose how they will proceed through the course and choose materials and activities that are appropriate and accessible for their students. Being able to supplement the content with a wide range of videos, technology, strategies, types of activities, etc also provides opportunities for teachers to differentiate and provide alternative approaches or interpretations on concepts or topics being discussed.

Teaching differentiated to meet the needs of all learners

IB programmes promote equal access to the curriculum for all learners. Differentiation entails planning for student differences through the use of a variety of teaching approaches, implementing a variety of learning activities and making a variety of formats and modes of exploring knowledge and understanding available to students. It also involves identifying, with each student, the most effective strategies to develop, pursue and achieve realistic and motivational learning goals. In the context of an IB education, special consideration often needs to be given to students' language backgrounds and skills. Affirming students' identity and valuing their prior knowledge are important aspects of treating students as unique individuals and helping them develop holistically as young people.

Assessment and mathematics

Both the internal and external assessment tasks of mathematics reflect the aims of the course. Being able to demonstrate critical thinking and problem solving is essential to mathematics and the assessments are designed to facilitate this. The internal assessment exploration is an opportunity for students to demonstrate their understanding and insights into an area of mathematics that is of interest to them, and to participate in an activity which gives them an insight into what it means to be a mathematician.

Students should be introduced to the assessment criteria of the course early on and these should be regularly referred to in terms of the skills being developed throughout the learning process. Students must have a clear understanding of how they will be assessed, and the expectations of the course.

Teacher feedback is crucial and should allow students to monitor their progress and reflect upon their learning and skills development.

Teaching informed by assessment (formative and summative)

Assessment plays a crucial role in IB programmes in supporting and measuring learning. Formal Diploma Programme assessments are based on course aims and objectives and, therefore, effective guidance of these requirements also ensures effective teaching. Formative assessments developed by teachers are tools and processes to improve student learning. Here, feedback is most effective as a two-way process: students learn how they are doing and teachers learn what students understand, struggle with, and find engaging. In addition to assessment tasks, such feedback can be provided more informally.

Approaches to learning

Thinking skills and mathematics

Thinking skills, and particularly critical thinking, are developed and practised continuously in mathematics; students are challenged to apply their knowledge and skills to unfamiliar contexts or to abstract problems. Thinking skills are further developed through the emphasis in the teaching on conceptual understanding and making the links between different topics. Students of mathematics are encouraged to engage with the approaches or interpretations of problems critically.

Thinking skills

IB programmes pride themselves on giving students opportunities to develop their thinking skills and an awareness of themselves as thinkers and learners. Being "thinkers" is one of the IB learner profile attributes, and is defined in terms of exercising initiative in applying thinking skills critically and creatively to recognize and approach complex problems, and make reasoned, ethical decisions.

Thinking skills consist of a large number of related skills. In the Diploma Programme, particular emphasis is placed on skills such as metacognition, reflection, critical thinking, creative thinking, and transfer. Metacognition, or control over one's cognitive processes of learning, can be thought of as a foundation for developing other thinking skills. When practising metacognition, students think about the ways in which they process information, find patterns, and build conceptual understandings. Once they become aware that they are using a variety of techniques and strategies to perform even the most basic learning tasks, students can be encouraged to consider if there are more effective or efficient ways to achieve the same learning, try out these new ways and evaluate them. Similarly, reflection is a thinking skill that plays a critical role in improving learning. When practising reflection, students think about the success, value or otherwise of their learning. The Diploma Programme course aims, assessment objectives and assessment tasks place a premium on higher-order thinking skills, such as critical thinking, creative thinking and transfer.

Communication skills and mathematics

In mathematics, communication skills are practised in a number of different ways; as a subject it invites approaches to teaching that encourage dialogue and discussion, but also requires a reflective engagement with the way in which the mathematics is expressed both verbally and in writing. This discussion can reveal alternative perspectives to problem solving.

Communication skills

Communication skills are not only important in IB programmes but are also an essential part of a wider dynamic in the learning community: they help to form and maintain good relationships between students, and between students and adults. Furthermore, being able to communicate well contributes to the development of students' self confidence and enhances their future prospects, as communication skills are a critical ingredient of success in working life.

Communication skills consist of a cluster of different skills and forms of communication. The ability to listen and understand various spoken messages, to read and understand diverse written texts and other forms of media, and to respond clearly and convincingly in spoken, written and digital form are all part of how students engage with others in the world. Some of these forms of communication are independent of era and culture, but interacting in and with the digital space is a significant part of most students' communication and social interaction. Online activities, which are often collaborative in nature, present exciting opportunities for the development of students' communication skills.

Social skills and mathematics

There are strong correlations between social skills, affective skills and the ability to reflect. These all play a very important role in mathematics. Much of the content and skills students develop will provoke students to consider their own understandings and how these are expressed to others. Students should be challenged to think about the relationship between the learner profile attributes and mathematics. For example, how do the attributes of caring and being principled relate to what they are learning in mathematics? The structure of the DP mathematics guides provides students with opportunities to appreciate the contributions that other cultures have made to the understanding of mathematics. This, and the section in the guide referring to international-mindedness, can be used to provoke class discussions in which students will reflect on their own views and those of others.

Social skills

Closely related to, and perhaps even more important than, communication skills are social skills, whose importance in IB programmes has to do with the development of the learner as a whole and the value of a community for learning. A starting point for developing students' social skills is to acknowledge that people differ greatly in terms of their degree of introversion or extroversion and that these differences should be respected. Similarly, different cultures have different expectations of appropriate behaviours in social situations. To be able to understand the perspectives of others, to form good relationships and to regulate one's own emotions and behaviour are at the heart of many of the IB learner profile attributes and the IB's aspiration to develop internationally minded students. School, being such a formative community in young people's lives, can play a significant part in the development of their social and emotional skills.

Self-management skills and mathematics

IB learners also need to learn to persevere and be emotionally stable as individuals. Learning to manage themselves is important for students in a demanding educational programme like the Diploma Programme, as well as a highly desirable competency for their later studies and employment.

DP mathematics is a course based around problem solving, with students being required to use the inquiry and modelling cycles and often, within this framework, being required to develop their own strategies. As such, students often need to persevere when a solution is not immediately apparent. The IB promotes the development of educational opportunities for students which set challenging goals and help to develop their persistence to achieve them.

The DP mathematics courses encourage students to analyse and propose solutions to real-life problems. This is considered one of the challenges in the DP mathematics course. Self-management skills are required for students to learn to persevere through problem solving. The learners need to follow a sequential process to solve the problem which is outlined in the inquiry and modelling processes. First they have to understand the problem, then devise a plan, create possible solutions, and interpret their answers. Throughout the process students may struggle and may not be able to solve the problem. In that case it is very important that teachers foster a growth mindset with their students and emphasize that with effort, dedication and perseverance understanding can be achieved.

The internal assessment exploration task also requires students to plan and organize their time, to ensure they have appropriate research techniques and the tenacity to engage with mathematics, to reflect upon this and monitor their own progress. It is an important part of the course as it allows students to engage in authentic mathematical activity. The time allocated to the "toolkit" is designed to provide students with a variety of strategies to help support them through their study of the content and will be invaluable to their success in the internal assessment exploration.

Self-management skills

Self-management skills consist of organization skills, such as setting goals and managing time and tasks effectively, and affective skills, such as managing one's state of mind, motivation and resilience.

Like other learning skills, self-management skills can be modelled and practised. For Diploma Programme students, time management is often a particularly pertinent organization skill. Strategies for improving time management include breaking down assignments into achievable steps and timelining each step, planning revision and study plans for tests and examinations, and building study timetables. An important aspect of such strategies is not only what they factually achieve with students' use of time but that they give students a perception of greater control over their time.

Effective self-management skills, in turn, enable students to gain some control over their mood, their motivation, and their ability to deal with setbacks and difficulties. A school environment where students feel they have a degree of autonomy and self direction and where they do not need to get things right the first time, where challenging but not too difficult objectives are set, and even where psychological techniques such as mindfulness training are taught, can all support the development of students' effective skills.

Research skills and mathematics

Research skills in mathematics are closely aligned with approaches to learning, focused on conceptual understanding and inquiry. Throughout the mathematics course students engage with methods and concepts developed by others. However, the internal assessment exploration gives them the opportunity to demonstrate their own engagement and understanding of an area of mathematics of interest to them. The aim of this task is to give students the experience of **doing mathematics** and the opportunity to reflect on this practice.

Research skills

Research skills are a central element of the inquiry-based pedagogy of IB programmes. While good research skills have always been at the heart of academic endeavour, the availability of digital resources and the explosion in the amount of information easily accessible to students make the development of research skills a particularly pertinent part of today's education. Also, learning to work with academic integrity and respecting the intellectual contributions of others is an important aspect of learning in all IB programmes.

Fundamental research skills include formulating focused and precise research questions, appraising sources, recording, analysing, evaluating and synthesizing information, and presenting and evaluating results.

Additionally, research today requires much more validating, comparing and contrasting of available information, and narrowing down the volume of data into a manageable quantity with regard to being discerning about what is relevant. Though confident in browsing and communicating online, students often lack the information literacy skills they need for the kind of effective and self-directed research they are expected to do as part of their inquiries.

Cognitive academic language proficiency and mathematics

A framework for the use of cognitive academic language proficiency

IB students must become fluent in the academic language associated with each of the subjects they study so that they can fully engage and demonstrate their proficiency. Mathematics has its own language in which many everyday words have a different, and often much more precise, meaning. This language also incorporates mathematical symbols and depictions that need to be understood and interpreted. The precision is what allows mathematics to be a powerful way of knowing and forms the basis of thinking in mathematics. The agreed understanding of meanings among mathematicians globally allows them to communicate and collaborate and to make progress in their mathematical endeavours. In developing their mathematical understandings, students are developing their cognitive academic language proficiency (CALP).

The grid below is a framework to help teachers plan strategies for student CALP development as part of learning within mathematics.

Figure 1

A framework for planning CALP development

Cognitive Academic Language Proficiency	PEDAGOGY				
	Background knowledge (BK)	**Scaffolding for:**			**Extended CALP**
SKILLS	Activating and building up BK	New comprehensible input	Processing of new input	New comprehensible output	Demonstrating and applying
Listening					
Speaking					
Interacting					
Reading					
Writing					
Thinking					

Understanding the framework for planning cognitive academic language proficiency development

This framework is organized as a grid. The component skills of CALP (including thinking skills, which support academic language proficiency) are set out in rows and the pedagogy is set out in columns.

Activating background knowledge

Background knowledge is the existing knowledge a student has in terms of the language of the subject. This may be from a previous course and could be in a different language altogether. When this is activated it provides a base for new learning.

Scaffolding and practice

Scaffolding is a strategy that enables learners to build on their background knowledge to extend their learning so they can accomplish more difficult tasks. Scaffolding activities allow for contextualization so new learning input is meaningful. New learning is fully acquired through practice.

Demonstrating cognitive academic language proficiency

Independently demonstrating and applying new CALP in novel and varied situations is a sign of successful learning. This new learning will become part of a student's background knowledge upon which more new and extended learning can be built during the next cycle.

Using the framework for planning cognitive academic language proficiency development

It is not expected that every single box on the framework will be completed in detail in each case. A lesson is often likely to focus on only some skills and aspects of pedagogy. However, over a period of time or a series of lessons, it would be sound practice to ensure that all the dimensions have been adequately addressed.

Additional pedagogy: Affirming identity

Affirming student identity is a central underpinning pedagogical principle for successful learning in which the activities for developing CALP are embedded. Affirming identity includes explicitly valuing students' skills and knowledge in all their languages and recognizing these as resources for teaching and learning new ways of thinking and knowing.

The following activities have been designed to develop CALP in mathematics.

Working with numbers written in the form $a \times 10^k$ where $1 \leq a < 10$ and k is an integer

The analysis of graphs

Working with vectors and equations of straight lines

Structuring the mathematics courses

There are numerous ways of structuring the Diploma Programme mathematics courses. The 60 hours of content common to the SL courses and the SL courses being a subset of the HL courses allows many different models of delivery to be considered. This allows schools to consider what will work in their own context and adopt the model which best suits their needs.

The mathematics guides, published in 2019 for first assessment in 2021, contain full details of the courses in terms of their nature and content. This TSM section on structuring the mathematics classes is designed to help teachers and schools think about the different ways that the classes could be structured.

The precise model used in a school could depend on many factors including:

- the needs of the students, their abilities, aspirations and motivations
- the resources available, including number of teachers and classrooms
- the skillset of the teachers
- the number of students taking a particular course
- timetabling and scheduling constraints
- the ratio of lessons for SL:HL is 5:8 (150 hours vs 240 hours).

When considering these factors, it is also worth keeping in mind that Mathematics: applications and interpretation at SL and HL will involve the extensive use of technology and it might be desirable for some lessons to have access to a suite of computers or other devices to support the teaching and learning of mathematics.

General considerations

Whichever model is used for the teaching of mathematics (considering the different approaches used in the two subjects) there will be opportunities for collaboration between groups of students studying different courses. The benefits of this are to enable students to appreciate the cohesive nature of mathematics and how their learning is interrelated. This might be while working on the internal assessment, for an inquiry-based lesson on the common content, or in applications of technology where students from one course can support students in another course.

Whichever model is used by schools, it is important that planning and reflection with the faculty/department takes place on a regular basis.

Depending on the teaching order of the topics, a school may allow students to move between SL and HL, or even between Mathematics: analysis and approaches, and Mathematics: applications and interpretation at an early point. It will be hoped that a number of students choosing to take an SL course at the beginning of their Diploma Programme studies, who then discover a passion for the type of mathematics they have chosen, will and can decide to take the HL course.

Model 1

Four separate classes where both Mathematics: analysis and approaches, and Mathematics: applications and interpretation HL are taught separately similarly for the two SL courses. All four courses can be taught throughout the two years of the Diploma Programme allowing for concurrency of learning.

A school could also adapt this model and offer three (or two) of the four courses.

Years 1 and 2	
Course 1	Mathematics: analysis and approaches SL

Years 1 and 2	
Course 2	Mathematics: analysis and approaches HL
Course 3	Mathematics: applications and interpretation SL
Course 4	Mathematics: applications and interpretation HL

Model 2

Both SL courses are taught as a subset of lessons of their respective HL courses. This means that for every eight scheduled lessons, SL students are present for five of these lessons alongside HL students. HL students have an additional three lessons on their own. This model would allow for both subjects to be offered at SL and HL using two teachers, although classes could be split and taught with more than one teacher.

Schools will need to take care in selecting topics, especially at the start of the course so that they are not introducing AHL material which requires the SL content to have been covered previously. The guides are not scope and sequence, or scheme of work documents. However, their structure guides makes clear the common content, the SL content and the AHL content.

Years 1 and 2	
Course 1	Mathematics: analysis and approaches SL (five lessons)
	Mathematics: analysis and approaches HL (three additional lessons)
Course 2	Mathematics: applications and interpretation SL (five lessons)
	Mathematics: applications and interpretation HL (three additional lessons)

Model 3

A combination of models 1 and 2 where a school may have three DP Mathematics teachers.

Mathematics: applications and interpretation is offered with SL and HL being taught separately throughout the two years, and Mathematics: analysis and approaches is taught as a combined class in the manner of model 2, or vice versa.

Years 1 and 2	
Course 1	Mathematics: applications and interpretation HL
Course 2	Mathematics: applications and interpretation SL
Course 3	Mathematics: analysis and approaches SL (five lessons)
	Mathematics: analysis and approaches HL (three additional lessons)

Incorporating the IB learner profile

DP mathematics courses at SL and HL are closely linked to and aim to engage students with the attributes of the IB learner profile. For example, the requirements of the internal assessment provide opportunities for students to develop every aspect of the profile. For each aim suggested, learner profile attributes are referenced below. Teachers are encouraged to discuss the interrelationship of the IB learner profile attributes and the aims of the mathematics course with their students. Some of the 10 learner profile attributes sit very easily with mathematics and students should be encouraged to think about those that do not immediately spring to mind when thinking about what it is to be a mathematician.

A discussion or activity related to this at the beginning and at points during the course can be a useful exercise for students to reflect upon their own development in terms of the learner profile attributes and as mathematicians.

Link to mathematics aims	Learner profile attributes
Develop a curiosity and enjoyment of mathematics, and appreciate its elegance and power	Inquirer
Develop an understanding of the concepts, principles and nature of mathematics	Knowledgeable
Communicate mathematics clearly, concisely and confidently in a variety of contexts	Communicator
Develop logical and creative thinking, and patience and persistence in problem solving to instil confidence in using mathematics	Thinker, balanced
Employ and refine their powers of abstraction and generalization	Reflective
Take action to apply and transfer skills to alternative situations, to other areas of knowledge and to future developments in their local and global communities	Open-minded, risk-takers
Appreciate how developments in technology and mathematics influence each other	Knowledgeable, reflective
Appreciate the moral, social and ethical questions arising from the work of mathematicians and the applications of mathematics	Principled, caring
Appreciate the universality of mathematics and its multicultural, international and historical perspectives	Open-minded
Appreciate the contribution of mathematics to other disciplines, and as a particular “area of knowledge” in the TOK course	Knowledgeable, balanced
Develop the ability to reflect critically upon their own work and the work of others	Reflective, communicator
Independently and collaboratively extend their understanding of mathematics	Inquirer

Connecting the content

Mind maps

The following resource will enable teachers and students to visualize the entire content of the mathematics courses and the connections between the various elements of those courses. This can be used in a variety of ways, for instance:

- a revision aid
- a way to introduce a topic and connect it to other parts of the course
- printed and posted in the classroom for reference
- projected and discussed to review a topic after the content has been covered
- used to find connections during classroom teaching.

Mind map

Supporting classroom activities

Time has been allocated within the teaching hours for students to undertake the types of activities that mathematicians in the real world undertake and to allow students time to develop the skill of thinking like a mathematician; in other words, providing students with a mathematical “toolkit” which will allow them to approach any type of mathematical problem. Underpinning this are the six pedagogical approaches to teaching and the five approaches to learning which support all IB programmes. This time gives students opportunities in the classroom for undertaking an inquiry-based approach and focusing on conceptual understanding of the content, developing their awareness of mathematics in local and global contexts, gives them opportunities for teamwork and collaboration as well as time to reflect upon their own learning of mathematics.

Students should be encouraged to actively identify skills that they might add to their personal mathematics “toolkit”. Teachers are encouraged to make explicit where these skills might transfer across areas of mathematics content and allow students to reflect upon where these skills transfer to other subjects the student is studying.

This section contains ideas and resources that teachers can use with their students to encourage the development of mathematical thinking skills. These resources have been developed by experienced teachers for use in their own classrooms. They have a content focus to give them a context but are not exhaustive.

The example activities have been designed to be used in three different ways. The first is that they could be used by teachers with their students as they stand, the second is that teachers could adapt the materials for their own context and the third is that they might inspire teachers to develop their own materials, perhaps using the same technique but with different content.

- Cognitive activators–engaging starting points
- Conceptual understandings–making use of the statements of conceptual understanding from the guide
- Using technology–some subject-specific examples of the ways in which technology can be used to teach certain skills or topics
- Modelling–an example of a modelling activity with notes as to why it is a good example
- Voronoi diagrams
- Differential equations, phase portraits and Euler’s method

Cognitive activators

Cognitive activators can be thought of as the strategies we use to get students ready to learn and engage with the subject material. Cognitive activators serve to introduce a new topic or concept and from there can lead into subsequent learning activities to acquire a certain skill or knowledge within a topic. They relate strongly to the activation phase of the cognitive proficiency in academic language. In this section several examples are provided to serve as examples of cognitive activators. They can be thought of as an implementation of the approaches to teaching which describe the key pedagogical principles that underpin the IB programme.

The strategies presented are based upon the ATL and recent research outcomes such as the Harvard visible-thinking project (see "Further reading for teachers and students").

As teachers we might not always be aware when we are (or are not) using cognitive activators when starting a new topic or lesson, but it is generally thought to be pedagogically sound to create some context or give some introduction to the students before launching into something new. Then, while studying the topic, we can better rely on our students to develop their inquiry and thinking skills to learn mathematics, rather than being instructed on a specific mathematical procedure. Being explicit about this is in itself a strategy that can help students to engage in learning experiences and become more self-managed in their approach to learning. Cognitive activation is thus meant to activate students, engaging them in the topic, and preparing them to discover the new material.

The following is a selection of strategies used by experienced IB teachers in their mathematics classes. These are aimed at inspiring and guiding teachers to adapt and use them in their own classrooms. Some of the common elements of cognitive activators are:

- connecting to previous learning on the topic or different topics within the same concept (for example, to introduce the "average rate of change" one could activate students by reminding them of the concept of "change", the slope of a line or increasing versus decreasing)
- starting with an essential question that can be understood by the students but only successfully answered by the new learning material. This can be supported by, for example, a random group generator, no-hands-up activity, think pair share etc. to organise collaboration.
- an activity that enables students to start their work on the topic
- ideas for reflection and/or extension to further topics.

In essence, cognitive activation is about teaching students the strategies that encourage them to think more deeply in order to find solutions and to focus on the method they use to reach the answer rather than simply focusing on the answer itself. As such it is a useful introduction to the skills needed for the internal assessment exploration. When considering other resources within this toolkit teachers and students will recognize elements of cognitive activation within them.

Cognitive activation has been identified as one of several practices that support the development of mathematical literacy.

Saving for a college fund–annuities and amortization

Fractals–matrices and transformations

Getting a feel for data

Getting a feel for data—student dataset

Anscombe's quartet

Anscombe's quartet–dataset

Conceptual understandings

The aim of this section of the TSM is to facilitate the use of the mathematics concepts by teachers to enhance learning within the classroom and therefore promote deeper understanding.

Concepts are important because they increase mathematical understanding and allow students to make connections and generalizations which are key in problem solving. This in turn makes students less reliant on learned techniques and structures and more able to think creatively when faced with more complex problems.

The concepts

The DP mathematics courses identify 12 fundamental concepts as shown below.

These can inform units of work and can help to organize teaching and learning. Explanations of each of these concepts in a mathematical context have also been provided. Teachers may identify and develop additional concepts as required by local circumstances and national or state curriculums.

Approximation	This concept refers to a quantity or a representation which is nearly but not exactly correct.
Change	This concept refers to a variation in size, amount or behaviour.
Equivalence	This concept refers to the state of being identically equal or interchangeable, applied to statements, quantities or expressions.
Generalization	This concept refers to a general statement made on the basis of specific examples.
Modelling	This concept refers to the way in which mathematics can be used to represent the real world.
Patterns	This concept refers to the underlying order, regularity or predictability of the elements of a mathematical system.
Quantity	This concept refers to an amount or number.
Relationships	This concept refers to the connection between quantities, properties or concepts; these connections may be expressed as models, rules or statements. Relationships provide opportunities for students to explore patterns in the world around them.
Representation	This concept refers to using words, formulae, diagrams, tables, charts, graphs and models to represent mathematical information.
Space	This concept refers to the frame of geometrical dimensions describing an entity.
Systems	This concept refers to groups of interrelated elements.
Validity	This concept refers to using well-founded, logical mathematics to come to a true and accurate conclusion or a reasonable interpretation of results.

Ideas about how to use the concepts

The concepts can be used in many different ways and teachers can decide on approaches appropriate to their own classes and contexts. The 12 concepts need not be used in each topic but should be developed across the whole curriculum as and when appropriate.

Different concepts provide different approaches to learning a topic and also enable links to be made within topics, across topics and to other subject areas.

Each topic in the guide begins by stating the essential understanding(s) of the topic, and gives some **suggested** content-specific statements of understanding. Teachers are encouraged to develop their own statements.

These tasks illustrate how the concepts and teaching for conceptual understanding can be implemented.

Triangular tangles–SL applications
The power of matrices–HL applications

Using technology

The use of technology is an integral part of DP mathematics courses. Developing an appreciation of how developments in technology and mathematics have influenced each other is one of the aims of the courses and using technology accurately, appropriately and efficiently both to explore new ideas and to solve problems is one of the assessment objectives. Learning how to use different forms of technology is an important skill in mathematics and time has been allowed in each topic of the syllabus and through the "toolkit" in order to do this.

Technology is a powerful tool in mathematics and in recent years increased student and teacher access to this technology has supported and advanced the teaching and learning of mathematics. Discerning use of technology can make more mathematics accessible and motivating to a greater number of students.

Teachers can use technology to support and enhance student understanding in many ways including:

- to bring out teaching points
- to address misconceptions
- to aid visualisation
- to enhance understanding of concepts that would otherwise be restricted by lengthy numerical calculations or algebraic manipulation
- to support students in making conjectures and checking generalizations
- to explicitly make the links between different mathematical representations or approaches.

Students can also use technology to engage with the learning process in many ways, including the following:

- to develop and enhance their own personal conceptual understanding
- to search for patterns
- to test conjectures or generalizations
- to justify interpretations
- to collaborate on project-based work
- to help organize and analyse data.

In the classroom teachers and students can use technology working individually or collaboratively to explore mathematical concepts. Key to successful learning of mathematics with technology is the fine balance between the teacher and student use of technology, with carefully chosen use of technology to support the understanding and the communication of the mathematics itself.

Many topics within the DP mathematics courses lend themselves to the use of technology. Graphical calculators, dynamic graphing software, spreadsheets, simulations, apps, dynamic geometry software and interactive whiteboard software are just a few of the many kinds of technology available to support the teaching and learning of mathematics.

Within the guide the term "technology" is used for any form of calculator, hardware or software that may be available in the classroom. The terms "analysis" and "analytic approach" are generally used in the guide to indicate an algebraic approach that may not require the use of technology. It is important to note there will be restrictions on which technology may be used in examinations, which will be detailed in relevant documents.

Finance packages on the graphical display calculator

Graphical calculators have functionality that allows finance calculations to be carried out with ease. The following unit plans illustrate two different scenarios which might arise in the course.

Financial applications of geometric sequences and series

Amortization and annuities

Monte Carlo simulations

Modern technology allows us to answer many questions to a high degree of accuracy, even if we cannot find the exact solution. One extremely powerful modern method of doing this is called Monte Carlo simulation, where we use random numbers to generate many possible sets of data and use this to investigate the question.

Although computers are an important part of the simulation process there are still three key skills which students need to develop. In many situations, these are far more important mathematical skills than the analytic methods traditionally emphasized in mathematics teaching.

- Describing the situation mathematically
- Converting this description into a computer simulation
- Interpreting the results of the simulation, including realising when a simulation fails

The following two classic examples of simulation should give students a valuable insight into modern mathematics and allow them to develop these skills. Sample datasets are available here as well, which can be adapted or used as a model for creating original datasets.

Shooting arrows at a target

Shooting arrows at a target–dataset

Overloading lifts

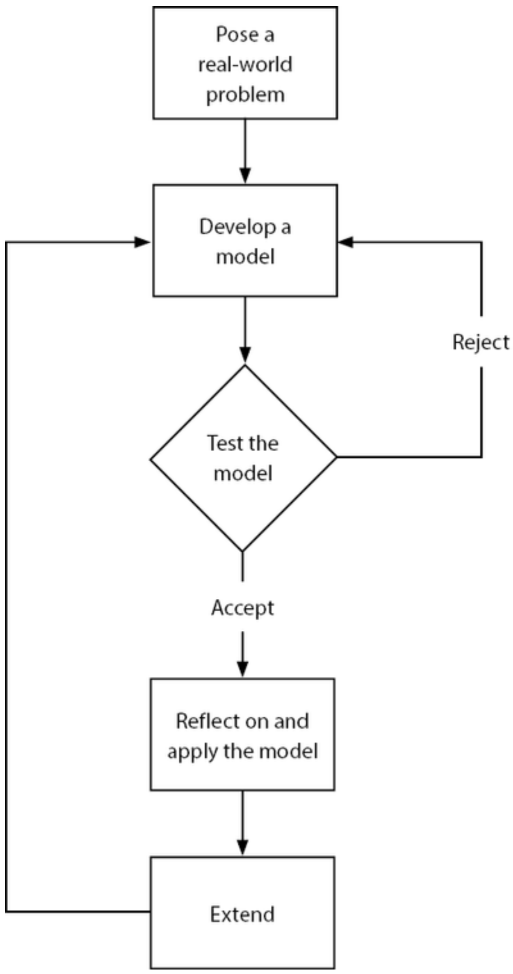
Overloading lifts–dataset

Mathematical modelling

Modelling is an important skill in mathematics and is becoming increasingly important as technology expands the bounds of what can and cannot be modelled. The guide gives an introduction and overview of the modelling process. Here we give one example of a modelling activity which has been designed to bring out the most important stages of modelling and act as an example of what the modelling cycle below looks like when this type of activity is carried out in the classroom.

The guide defines a range of different functions to be used for modelling and guidance is given as to the different contexts where these functions could arise. There are many resources with data available online, however it is often more engaging for students to gather their own data if the context allows this.

The cycle of mathematical modelling is illustrated below.



The following activity seeks to bring out the important elements of mathematical modelling.

Poachers on the game reserve–student activity

Poachers on the game reserve–teacher notes

Voronoi diagrams

Voronoi diagrams are being used increasingly widely and are a very authentic use of a mathematics application. They have many practical uses in ecology, epidemiology, urban planning, deliveries, service areas, control of robots, rovers and driverless cars, and in graphic design.

Voronoi diagrams partition space and can be used for questions such as these.

- How can we accurately map the territories of animals to prevent overcrowding?
- Where's the best place to open a new restaurant to steal competition from another restaurant?
- If a city has several hospitals with a helicopter, what's the service area of each helicopter?
- If I know how much it has rained in several locations, how can I estimate rainfall in other nearby locations?

Voronoi diagrams are a rich application of mathematics and are becoming of more and more of interest to mathematicians as technology has developed to support their creation. They are a fine example of how developments in technology and mathematics influence each other (aim 7). This resource aims to introduce teachers and students to the concept of the diagram, how they are constructed and two important techniques connected to the Voronoi diagram–the incremental algorithm and the nearest neighbour interpolation. There are also six learning activities which teachers may use directly or adapt for their own uses.

What's a Voronoi diagram?

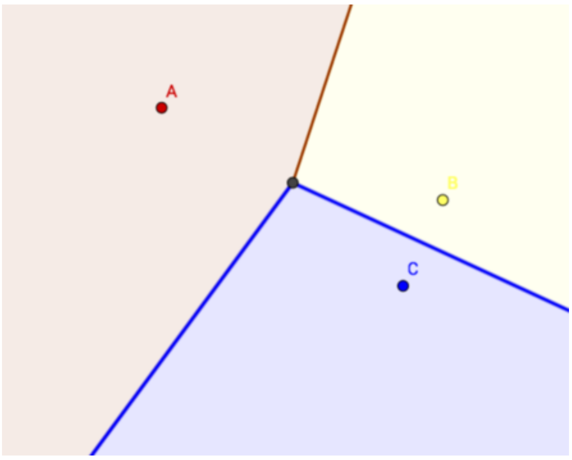
Suppose we are given a set of **sites** (points) in a bounded or unbounded plane (Fig. 1). A **Voronoi diagram** answers this question: which points are closest to point A? To point B? To point C? The Voronoi diagram for this set of sites divides the plane into Voronoi **cells or regions** (polygons). Each cell contains all the points in the plane that are closer to that site than any other (Fig 2). The line segments dividing cells are **edges or boundaries** and intersection points of edges are **vertices**.

Figure 1

Sites A, B, C

Figure 2

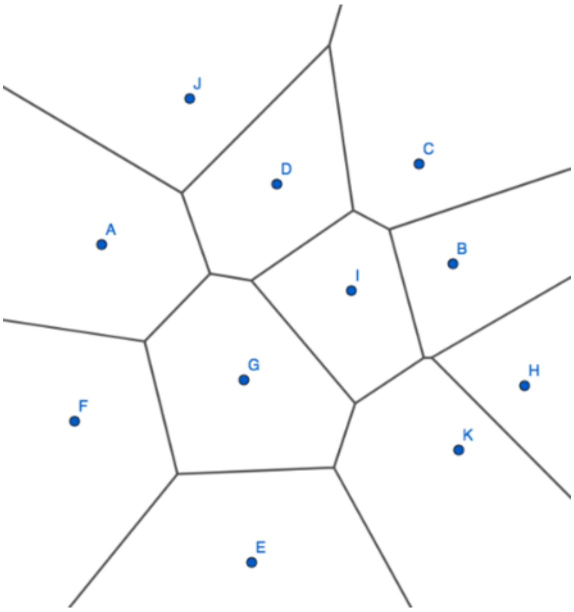
Cells for sites A, B, C



Now, given any point in the plane, we can determine which site it is closest to. Any point in the blue region, for example, is closer to site C than to sites A or B.

A more complex example is shown in Figure 3:

Figure 3



Constructing a Voronoi diagram

There are many algorithms for constructing Voronoi diagrams. The one demonstrated here is an **incremental algorithm** that builds the diagram recursively, adding one site at a time.

To understand it, we first look at the structure of a Voronoi diagram. Because each edge of a Voronoi diagram is equidistant to two sites, it must lie on the perpendicular bisector of the segment joining those sites (Fig 4). This implies that each vertex of the Voronoi diagram is a circumcentre of a triangle formed by a set of three sites (Fig 5).

Figure 4

Edges lie on perpendicular bisectors of AB, BC, AC

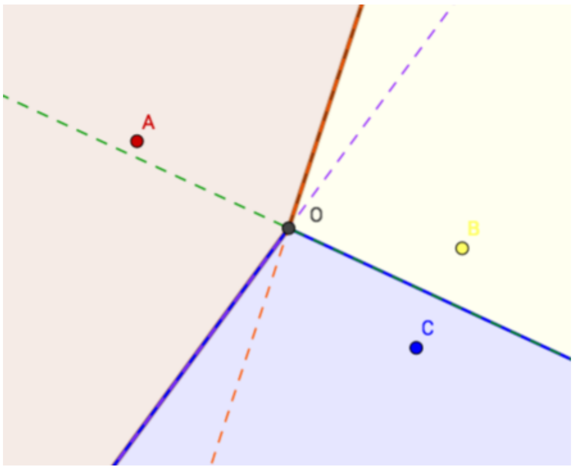
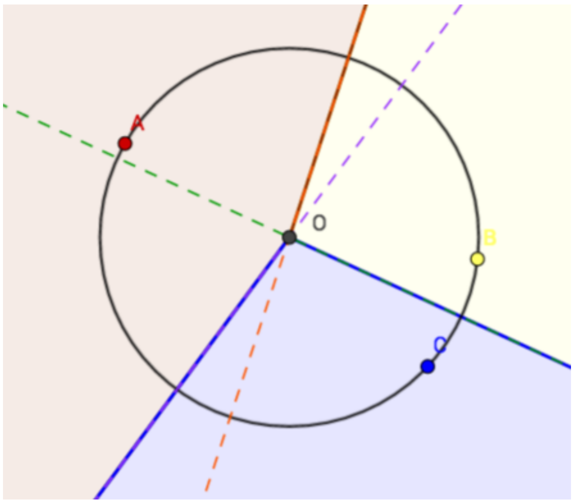


Figure 5

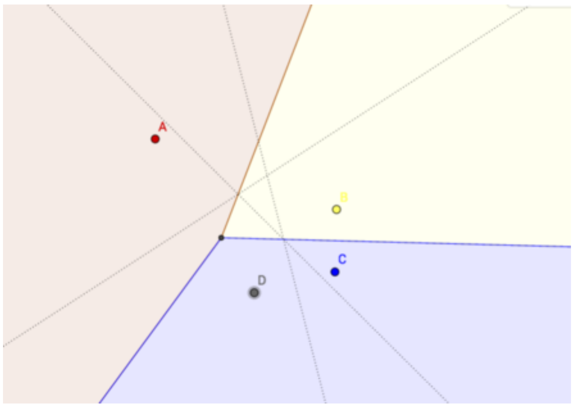
Vertex O is the circumcentre of triangle ABC



The incremental algorithm utilizes these relationships to build the diagram recursively, one site at a time. Given the diagram above, suppose we wish to add a fourth site, D. We draw the perpendicular bisectors with all nearby sites, $\underline{AD}$, $\underline{BD}$, and $\underline{CD}$, as shown in Figure 6.

Figure 6

Adding a fourth site D and its associated perpendicular bisectors

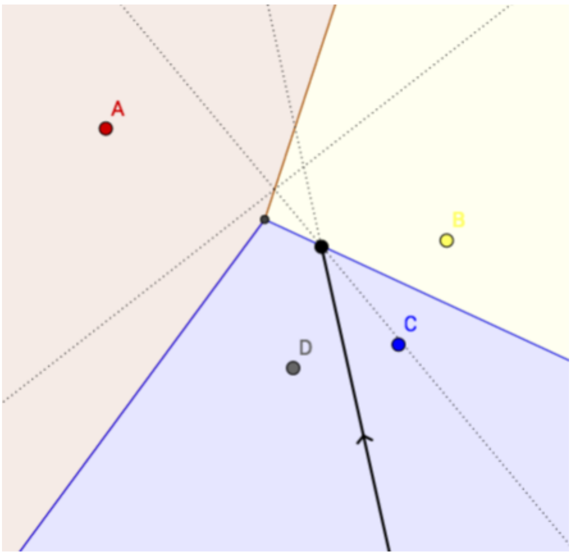


Travel along the perpendicular bisectors to create the new cell:

1. begin with the bisector of $\overline{CD}$, because D is inside cell C
2. follow this bisector in either direction toward an edge. In this example, we reach the edge between sites B and C

Figure 7

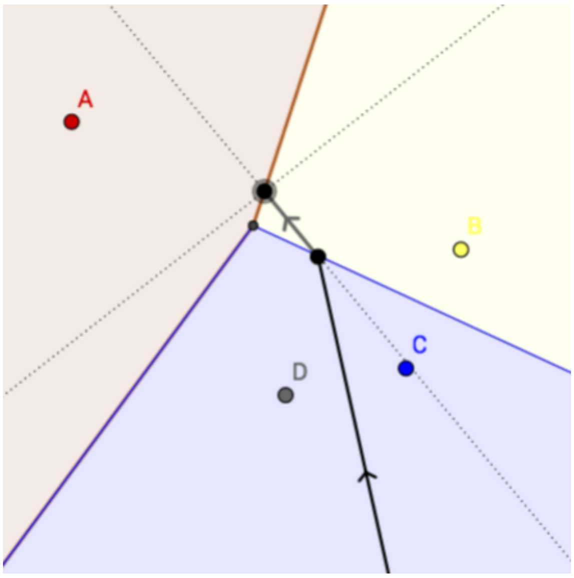
Incremental algorithm, steps 1 and 2



1. follow the perpendicular bisector of site D and the new site (in this case, B) until another edge is reached

Figure 8

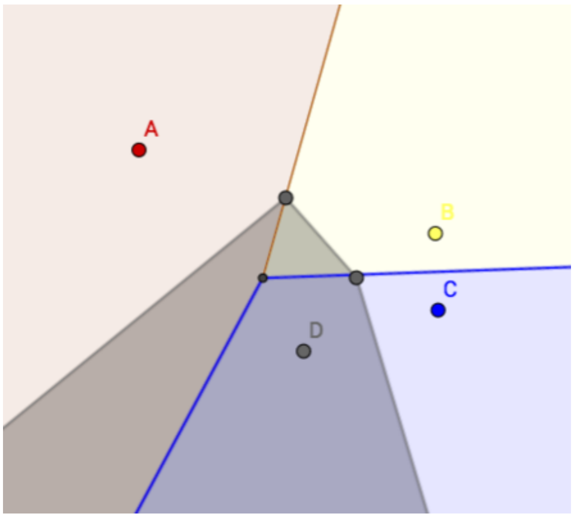
Incremental algorithm, step 3



1. repeat until you either intersect the boundary of the diagram or return to site D. If you hit a boundary, return to site D and start the same process but in the opposite direction
2. the segments you've travelled along are the edges of the new Voronoi cell for site D (the shaded area).

Figure 9

Incremental Algorithm, steps 4 and 5

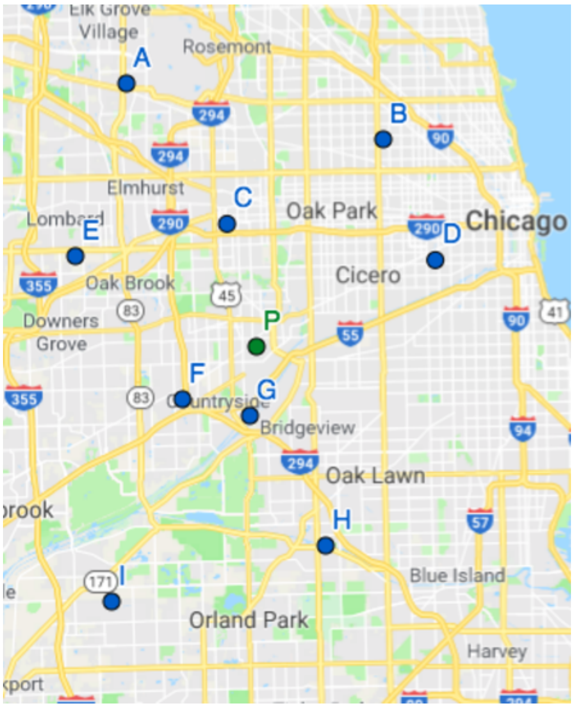


Nearest neighbour interpolation

An important application of Voronoi diagrams is to interpolate values of a function at points near sites, given its value at those sites. For example, suppose we measure lead concentration levels at various sites:

Site	A	B	C	D	E	F	G	H	I

Lead concentration (ppm)	1140	970	1365	525	350	680	310	120	70



How can we estimate concentration levels at another point? The simplest method is called **nearest neighbour interpolation:** determine the cell to which the point belongs, and assign it the same function value as the cell's site. That is, if point X lies in the cell of site S_1, then we estimate that $f(X) = fS_1$ for a function f that assigns a real-number value to points in the diagram.

Uses and contexts

Voronoi diagrams can be used to answer three broad categories of questions.

1. **Distance:** Which points are equidistant to several sites? What's the furthest distance from any site in the diagram? What's a path that stays as far as possible away from the sites? Where can we locate something so that it is as far as possible from existing sites?
2. **Area:** What are the ”territories” of animals/restaurants/helicopters–regions of influence? Where can we put a new site so that it can be used by the most/least current areas?
3. **Function interpolation:** Estimate values at different locations; find an average value over a whole diagram.

They appear in a variety of contexts, including:

- ecology–sites are, for example, watering holes where animals congregate; cells are areas that depend on that water source
- epidemiology–spread of disease from contaminated sites (this was an original use of the diagrams; see article in "Further reading")
- urban planning–sites are a resource such as schools, police, firemen; cells are areas that utilize that resource
- deliveries–sites are warehouses; cells are delivery areas
- service area of Craigslist sites
- robots/rovers taking a path to avoid certain objects on a factory floor/planet

- Colouring graphics–sites are specific colours; cells are areas of similar colour; natural neighbour interpolation used to blend colours gradually from one site to another.

Other techniques that can be used:

- function interpolation–natural neighbour interpolation (more accurate)
- sweep line algorithm for constructing diagrams
- area calculations: Heron’s formula, Pick’s theorem
- using different metrics (for example, taxi-cab metric for fire stations).

Voronoi diagrams: Learning activities

Learning activity 1: Three ice cream stands

Key understanding: students will understand the definition of a Voronoi diagram by constructing one in a real-world context.

Time: 15 minutes.

Part 1: explore: if you are somewhere in North Boulder Park and want ice cream as fast as possible, which ice cream stall should you go to?

Desmos option: ask students to complete Desmos Activity: Ice Cream following the accompanying teacher notes.

Low-tech option

- Print out or project the map above and ask students to find the closest ice cream stand for several points.
- Copy the map onto transparencies and ask students to colour in the area closest to each ice cream stand.
- Collect students' sketches and overlay and display them.
- Compare and discuss areas of agreement and non-agreement in the sketches.

- Then ask students to reflect and discuss–how could we create a more accurate solution?

Part 2: Introduce the Voronoi diagram

- Show students the Voronoi diagram below—how well does it agree with their predictions?
- This diagram can be used to introduce basic vocabulary—sites, cells, edges, vertices.

Learning activity 2: Constructing simple Voronoi diagrams

Key understanding: students will understand how perpendicular bisectors can be systematically chosen to construct a small Voronoi diagram.

Time: 30–40 minutes.

Prior knowledge: students will need to be familiar with the concept of a perpendicular bisector and be able to use the GeoGebra tool or straight edge and compass to construct them.

Materials: GeoGebra Applet, internet access, and/or paper, compass and straight edge.

Differentiation: the entire activity can be modified to be more or less challenging by requiring only two points to be investigated or investigating four points. Students may also find it helpful to demonstrate the equal distance of the edges to the sites by modelling with people and string/measuring tape.

Investigation questions

1. Open the GeoGebra Voronoi App. Observe—how are the edges positioned relative to the sites? Be as specific as possible.
2. Move the sites A, B, and C around. Do the relationships you observe still hold? Be sure to test extreme cases.

3. Connect two sites with a line segment. How is the edge between these two sites related to this segment? Test your conjecture with other pairs of sites.
4. Describe a mathematical method for constructing your own Voronoi diagram of three sites.
5. Compare your method with another student's. Are your methods similar? What revisions, if any, do you want to make to yours?
6. On a blank GeoGebra template (or sheet of paper), plot three sites and construct their Voronoi diagram using perpendicular bisectors.
7. Test your Voronoi diagram's accuracy by choosing several points in different cells and measuring to verify that they are indeed closest to the site in their cell.
8. What point(s) will be equally distant to all sites? Justify your answer.
9. Optional: check your diagram by constructing the diagram using GeoGebra's "Voronoi" command.

Learning activity 3: Using the power of algorithms

Key understanding: students will understand the logic of an incremental algorithm for constructing larger Voronoi diagrams and use it to evaluate how adding a new site to an existing diagram affects distances and areas.

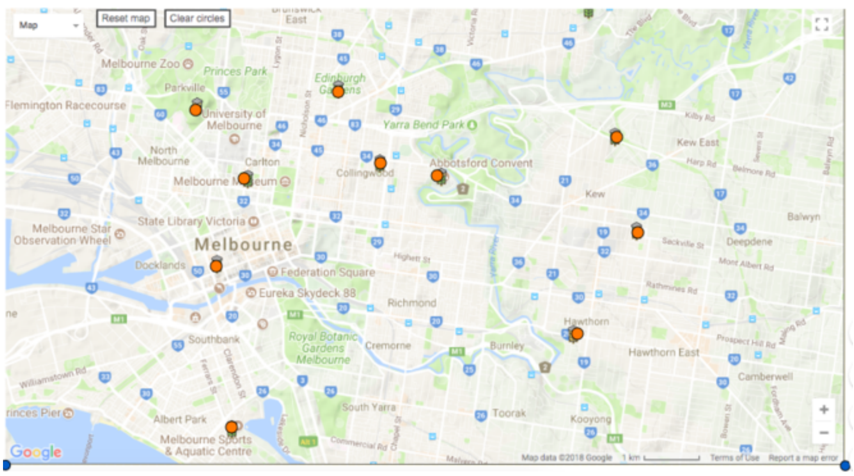
Time: one hour

Materials: copies of maps for students to sketch on; GeoGebra access.

Differentiation: extensions—Pick's Theorem for calculating the areas of polygons; use of taxicab metric instead of Euclidean metric to calculate distances.

Investigation: Beekeeping and bigger diagrams

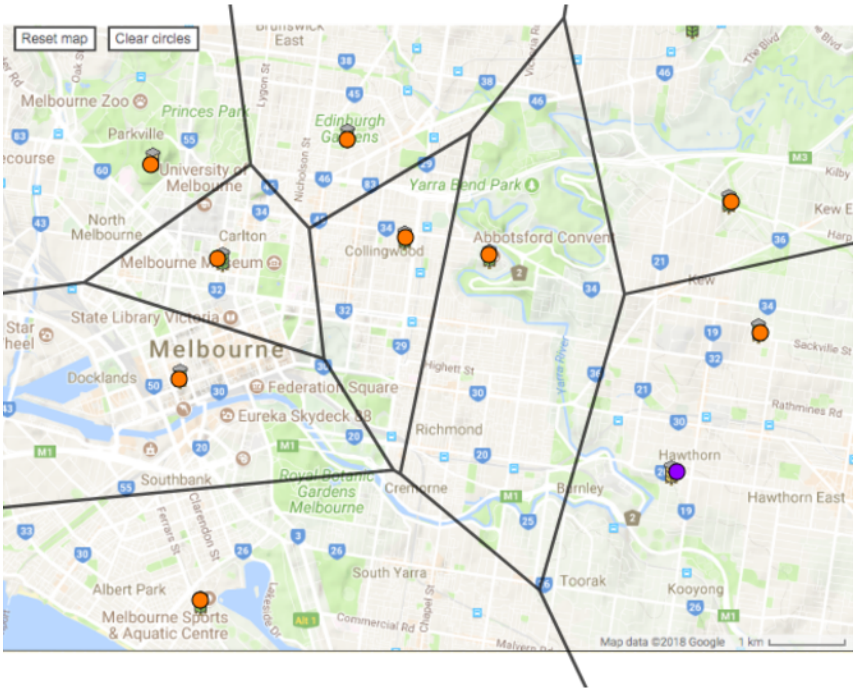
Beekeepers can share the location of their apiaries (clusters of beehives) on the internet. This can be useful to prevent overcrowding, as each beehive needs sufficient territory to survive. Below is a map of part of Melbourne, Australia with apiaries marked in orange. You can search for your apiary map on the internet.



Step 1: Which apiaries do you suspect have the smallest territories? Which have the largest? How would a Voronoi diagram help us answer this question more precisely?

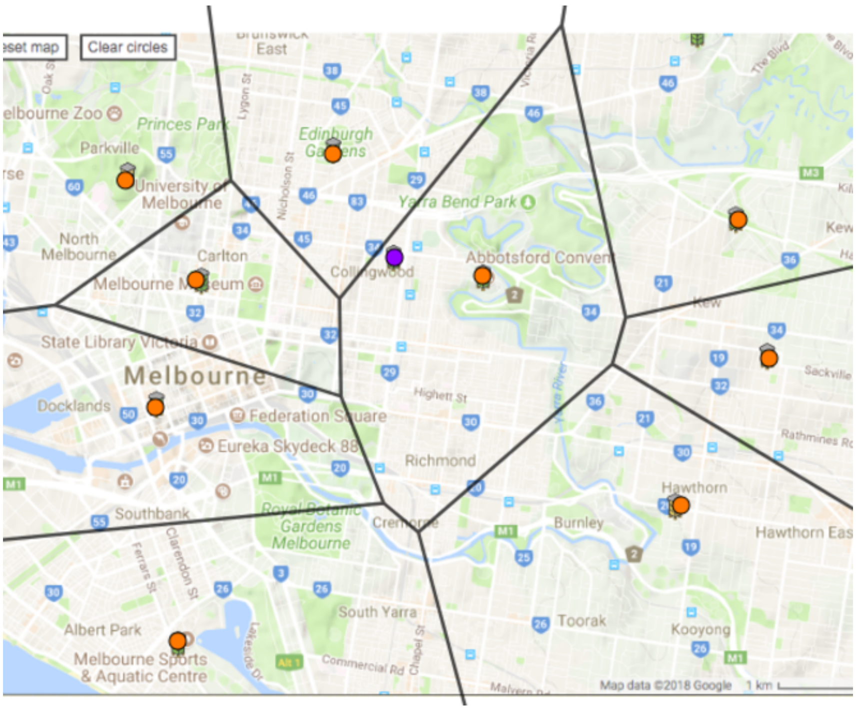
Step 2: Think about the method you've developed for constructing a three-site Voronoi diagram. What modifications or challenges do you anticipate in extending this to 10 sites?

As there are many sites in the diagram, it could be helpful to construct the diagram one site at a time. We call this process an *incremental algorithm*. To understand how it works, suppose that you have already drawn the diagram for nine of the 10 sites (apiaries) in orange.



Step 3: Based on your knowledge of the relationship between Voronoi edges and sites, sketch the new Voronoi cell for the 10th site, highlighted in purple, on your paper copy of this map.

Step 4: Now try sketching the cell if the 10th site is the one highlighted in purple.



Step 5: Next go through the incremental algorithm step-by-step to accurately construct the Voronoi cell for the 10th site that you just sketched. As you go through the animation, draw along on a fresh copy of the map. You may find it helpful to outline the updated diagram's edges with marker.

Step 6: Try performing the incremental algorithm on the other cell that you sketched above. You can check your answer with this diagram.

Step 7: Choose a location for an 11th apiary that you think would result in having the largest possible area for its cell. Use the incremental algorithm to add that 11th apiary to the 10-site map that you created in step five.

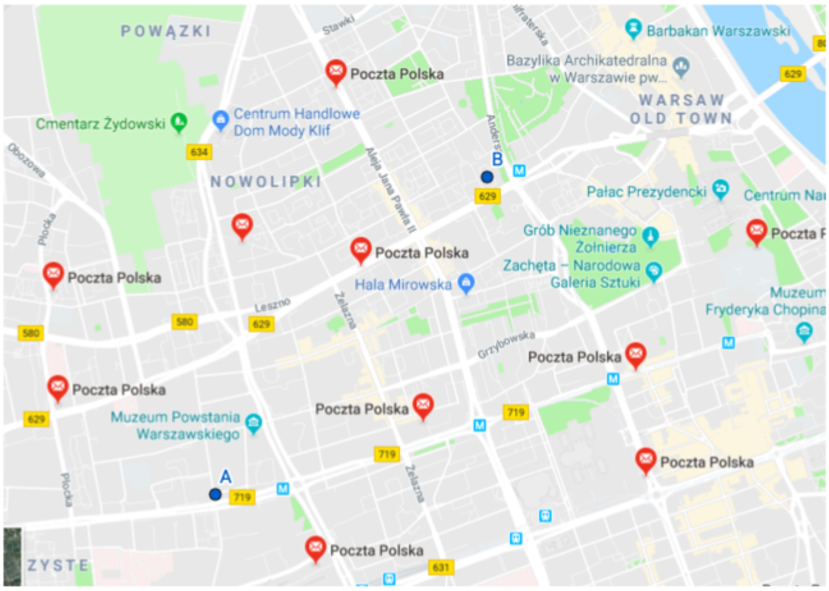
Step 8: Reflection questions

- Why does this method work? Are there any cases where it would not work? If so, can it be adjusted to work?
- Why do the perpendicular bisectors intersect each other at an edge of the existing Voronoi diagram?

Extension—bees thrive when their beehive has a radius of at least 1.5km from neighbouring beehives. Which apiaries in the diagram meet this requirement?

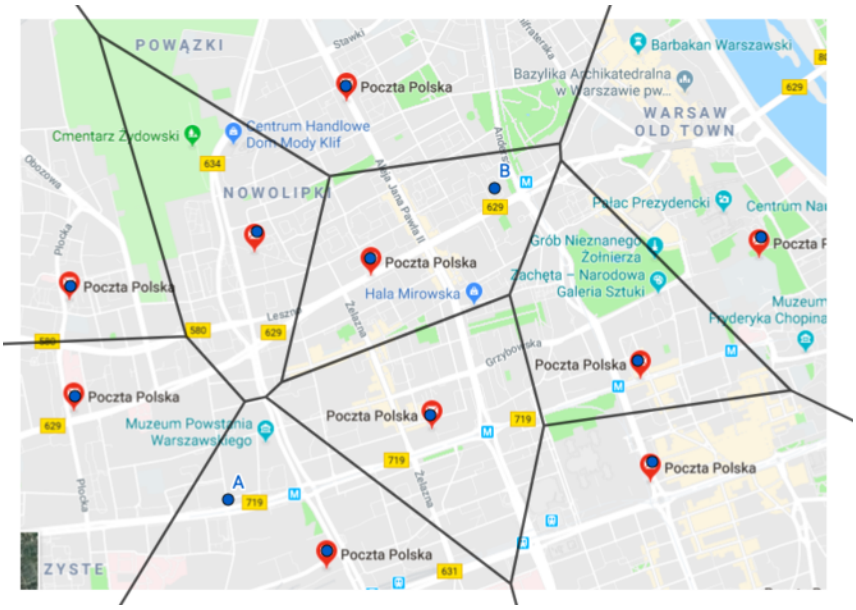
Application: Post office delivery areas

A new post office delivery centre is to be opened in Warsaw, Poland, and the government is considering two locations for it (labelled A and B) in the diagram.



Step 1: What relevant information might a Voronoi diagram provide about the post office locations?

Step 2: The government decides it would like to choose the location that will service a larger area. Below is the Voronoi diagram for the existing offices. Use the incremental algorithm to construct Voronoi cells for the two new sites.



Step 3: Based on your diagrams, estimate which location will provide a larger service area. You can check your diagrams and see exact areas of the two Voronoi cells with the solution here.

Learning activity 4: Predicting from Voronoi diagrams

Key understanding: students will understand that the data values at sites of Voronoi diagrams can be used to interpolate data values of nearby locations.

Time: 15–25 minutes.

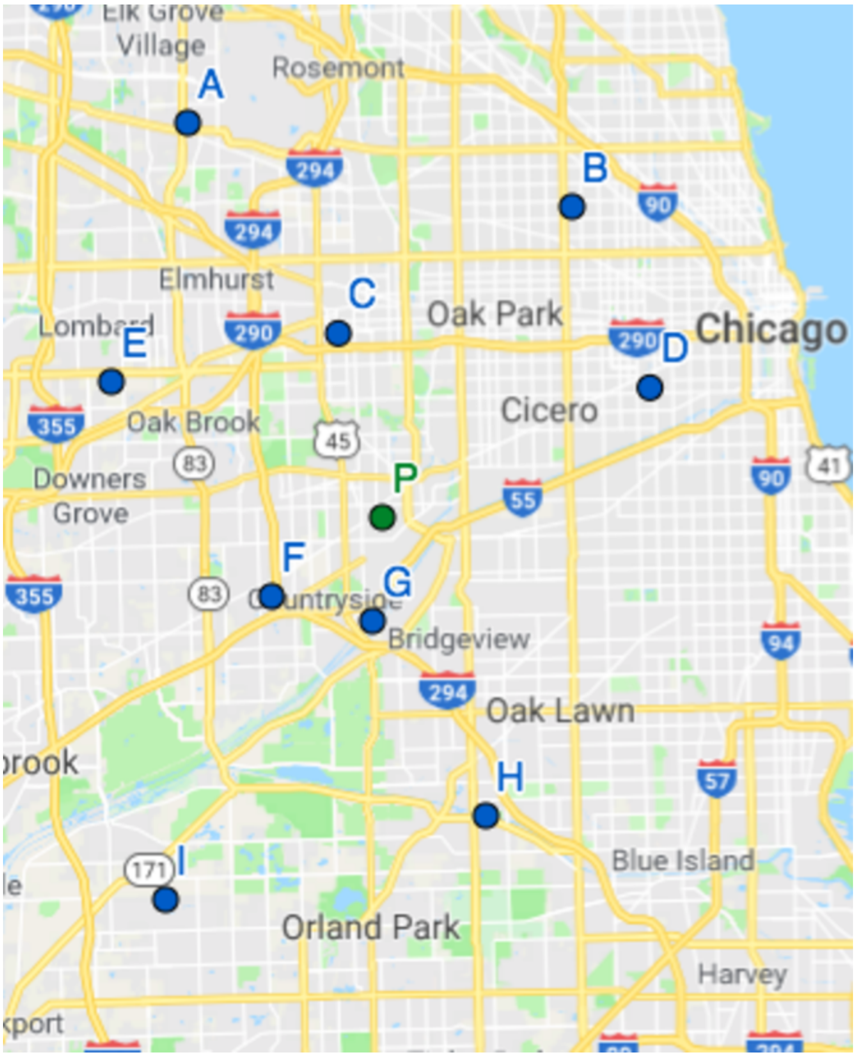
Differentiation: students can also apply the natural neighbour interpolation to more accurately calculate the pollution level. This is illustrated at the end.

Investigation: Predicting pollution levels

The element lead is found in soil and can be toxic to humans, especially children. The US Environmental Protection Agency states that the maximum safe lead concentration in soil is 1200 parts per million (ppm). However, for children's play areas, the maximum safe level is 400 ppm.

The EPA tests the lead concentration at several sites in the city of Chicago. Lead concentrations in ppm for each site are given in the table, corresponding to locations on the map.

Site	A	B	C	D	E	F	G	H	I
Lead concentration (ppm)	1140	970	1365	525	350	680	310	120	70



Step 1: Why might a Voronoi diagram provide useful information in this context?

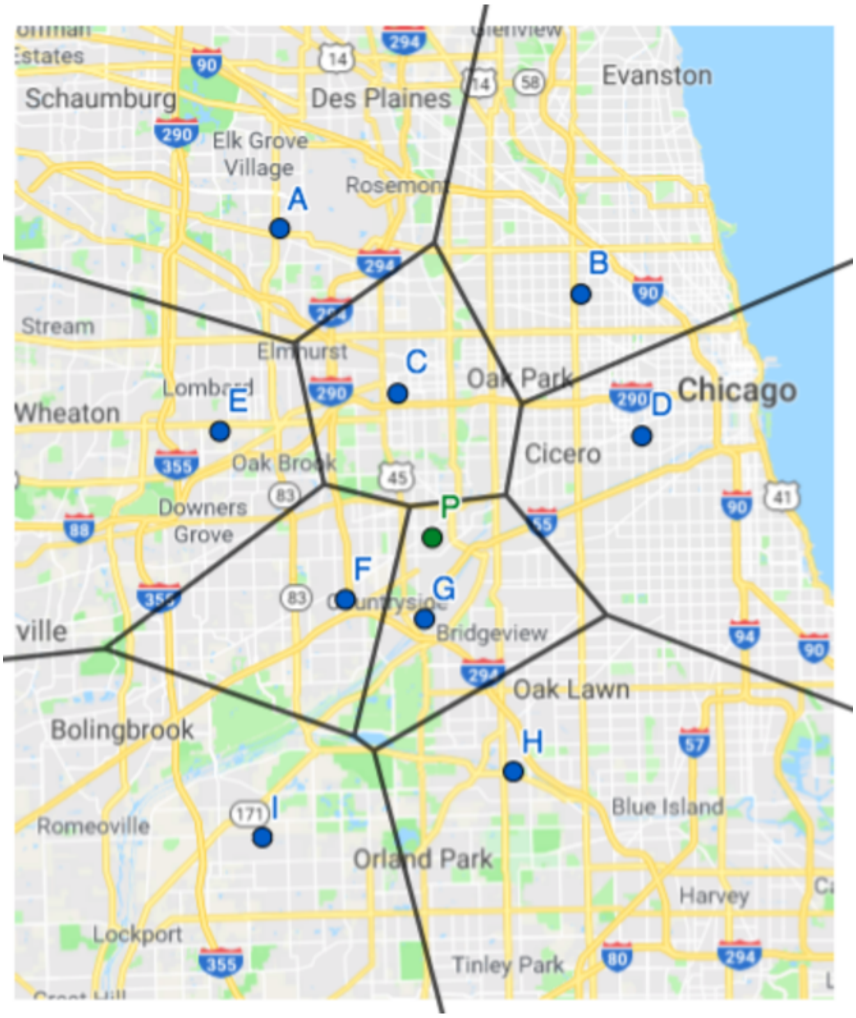
Step 2: Use GeoGebra to construct a Voronoi diagram of the sites.

Step 3: The city has proposed building a children's playground at point P. Estimate the lead concentration at this point, supporting your answer with reason(s).

Step 4: One simple way we can estimate the lead concentration at point P is by assuming it is the same as at site G, because point P belongs to cell G. This is called nearest neighbour interpolation. What are advantages to this interpolation method? What are disadvantages or sources of inaccuracy for this method?

Step 5: On a printout of your diagram, shade the locations that are unsafe for building a playground, according to the nearest neighbour interpolation.

Solutions



Point P is in the cell of site G, so its level is 310 ppm < 400 ppm. Hence it is safe.

It requires only drawing the Voronoi diagram but is not too accurate because it assumes the value stays the same within the entire cell.

Shade all cells with concentration > 400 ppm.

Extension: Natural neighbour interpolation

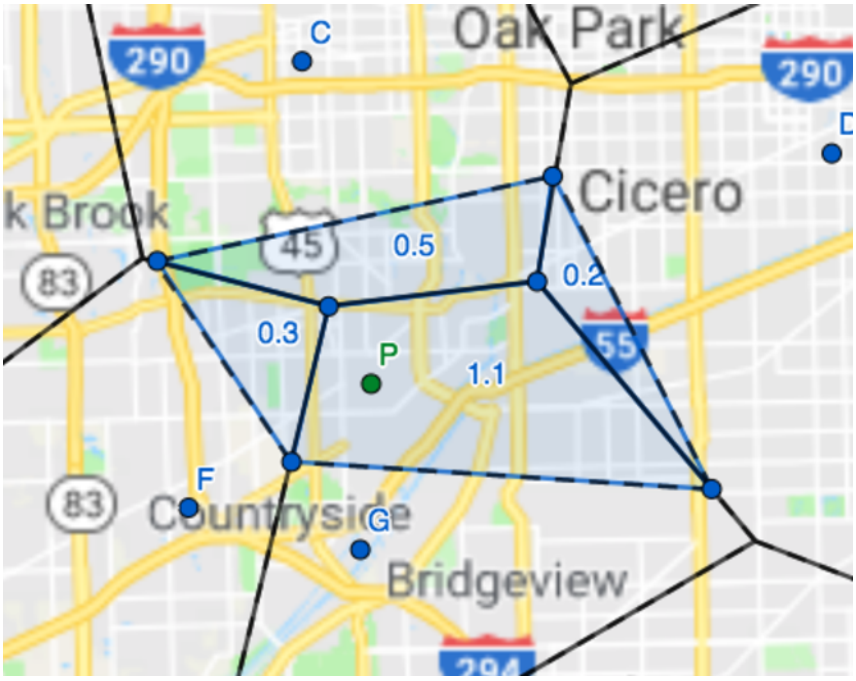
Step 1: On the diagram (by hand or in GeoGebra), draw and calculate the areas that would be needed to calculate the **natural** neighbour interpolation for point P.

Step 2: Find the natural neighbour interpolation of the lead concentration at point P. Based on this, is it safe to build a playground in this location?

Step 3: Which interpolation method provides the more accurate estimate of the playground's lead concentration: nearest neighbour or natural neighbour? Why?

Solution

Solution 1: Draw the Voronoi cell for site P (using the incremental method) on top of the original Voronoi diagram. Measure the area of each existing cell that the new Voronoi cell of site P would "steal" area from.



Solution 2: The interpolated value is an average of the values at relevant sites weighted by area "stolen" from those sites:

Site	A	B	C	D	E	F	G	H	I
Lead concentration (ppm)	1140	970	1365	525	350	680	310	120	70

$$f(p) = \frac{f(C)\times 0.5 + f(D)\times 0.2 + f(F)\times 0.3 + f(G)\times 1.1}{(0.5+0.2+0.3+1.1)}$$

$$= \frac{(1365\times 0.5 + 525\times 0.2 + 680\times 0.3 + 310\times 1.1)}{(0.5+0.2+0.3+1.1)}$$

$$= 635$$

Solution 3: As soil concentration is continuous, natural neighbour is a more accurate estimate. Hence, the soil is not safe for a playground.

Learning activity 5: Helicopter service in the coordinate plane

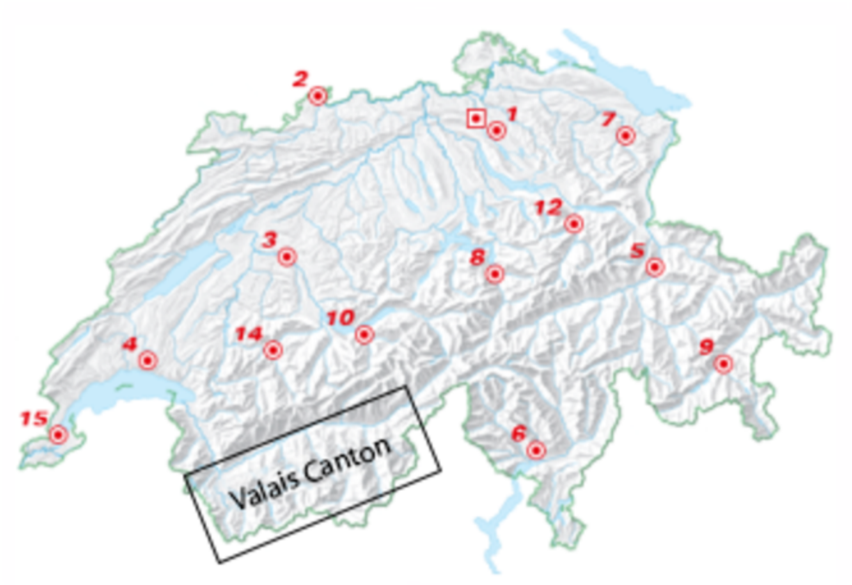
Key understanding: students can apply concepts of coordinate geometry to identify edges, vertices, and areas of Voronoi diagrams and interpret these in context.

Time: 20–30 minutes.

Materials: handout of Voronoi diagram with grid.

Application: Helicopter service

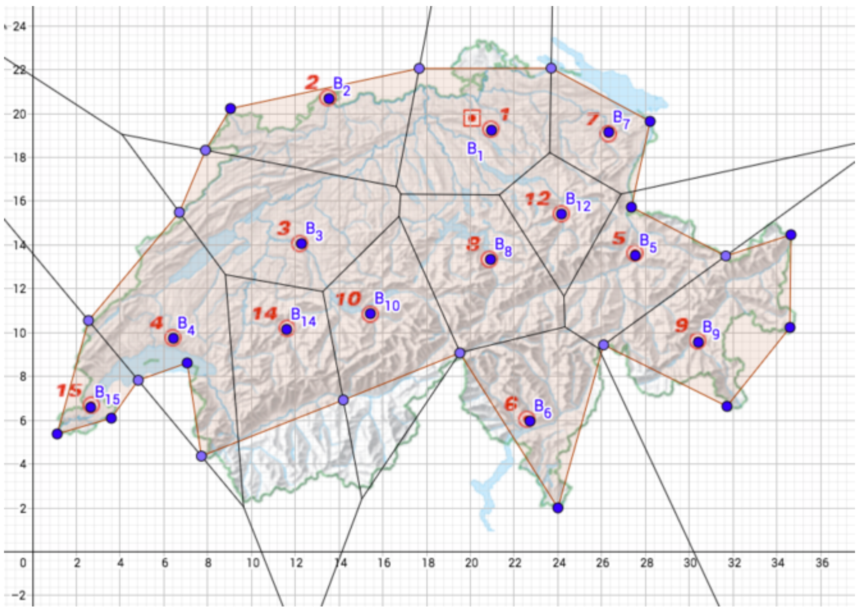
The Rega organization provides emergency helicopter services to the country of Switzerland, except for Valais Canton. It has 14 bases placed throughout the country, as shown in the map (https://www.rega.ch/en/our-missions/locations-and-infrastructure).



You are part of a consulting team that is evaluating the efficiency of the helicopter services.

Step 1: Your colleague suggests that a Voronoi diagram with the helicopter bases as sites would provide useful information about the service area of each helicopter. Explain whether you agree.

Here is a Voronoi diagram created in a coordinate plane with a scale of 1 unit = 10 km.



Step 2: On the diagram, identify the point or points that can be serviced equally quickly from bases three, four, or 14. Explain your reasoning.

Step 3: Given that the point furthest from any base lies in the Voronoi cell for base four, find this point and its distance from a base. Use the coordinate plane when calculating distances to increase the accuracy of your answer.

Step 4: When an emergency call is received, a base can get a helicopter flying within four minutes. Helicopters travel 400 km/hr on average. Rega's objective "is to be able to reach any location in

Switzerland–except in the canton of Valais–within 15 minutes' flying time." Can Rega reach its goal of responding to an accident anywhere in the country within 15 minutes? Justify your answer.

Solutions

Solution 2: The circle in blue is the only point equidistant from all three bases, as it is the intersection point of the three perpendicular bisectors of segments B_3B_4, B_3B_{10}, $B_{10}B_4$.

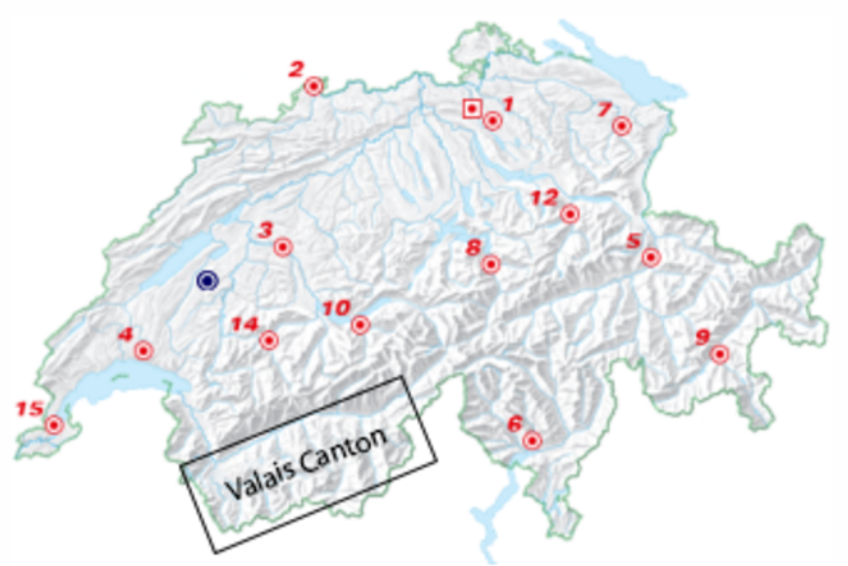
Solution 3: Use distance formula to calculate distance to each vertex you (can exclude ones that are obviously not the farthest). Farthest is the site directly north of the site, distance 5.76 units = 57.6km

Solution 4: All locations are within 58km of a base, as this point is the furthest in the entire diagram.

At 400 km/hr, 58km takes 0.145 hrs = 8.7 min

Total time = 8.7 min < 15 min

Yes, it can.



Learning activity 6: Interpolating precipitation in the coordinate plane

Key understanding: students can apply concepts of coordinate geometry to identify edges, vertices, and areas of Voronoi diagrams and interpret these in context.

Time: 1.5–3 hours.

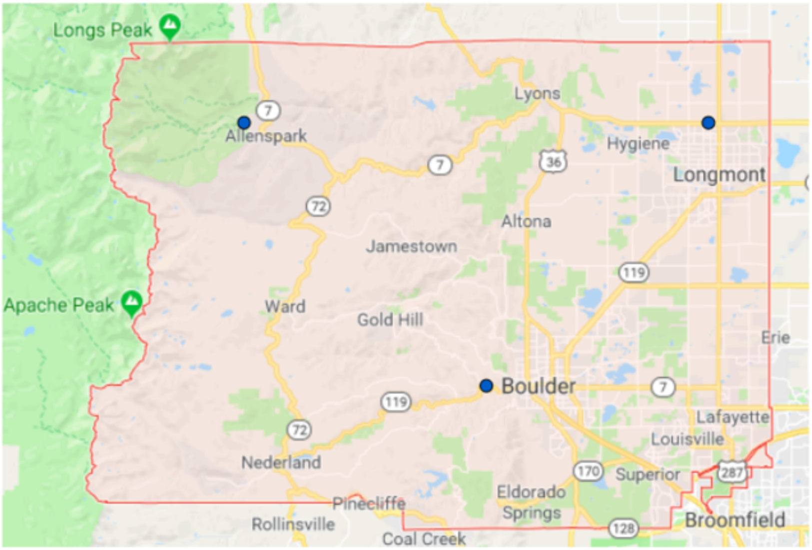
Materials: copies of diagram with grid to draw on.

Differentiation: using GeoGebra to find coordinates and/or areas will lessen the algebraic difficulty of the task.

Application: Precipitation in Boulder County

Note: throughout this activity, round your answers appropriately so that your final solutions are accurate to one decimal place.

Meteorologists in Boulder County, Colorado in the US track the cm of precipitation (rainfall and snowfall) received at three sites, marked on the map below:



Step 1a: Which station would best predict the precipitation of Lyons? Nederland? Jamestown? Explain your reasoning.

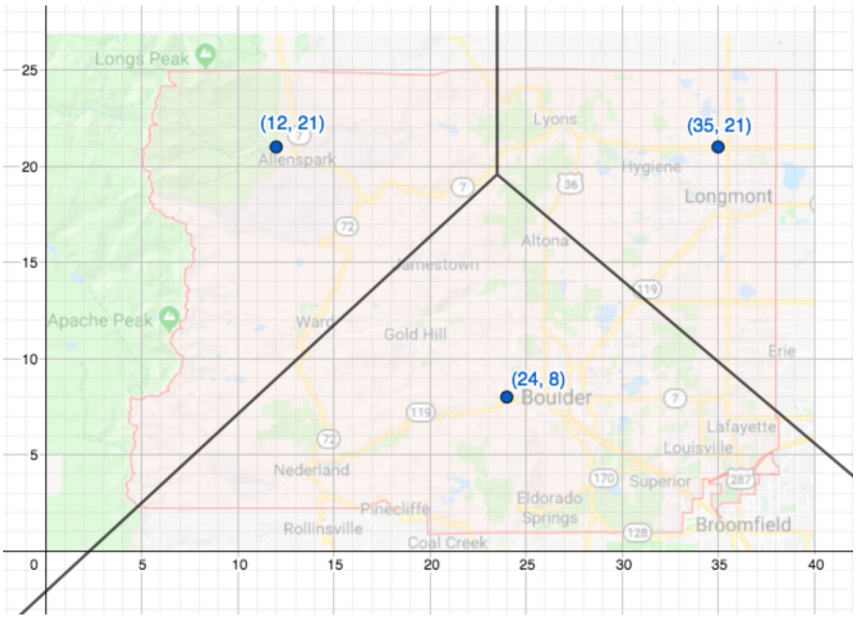
Step 1b: What information would a Voronoi diagram give you about this situation?

A Voronoi diagram of the three precipitation collection sites is now drawn on the grid below, with a scale of 1 unit = 1.6km for both axes.

Step 2a: Use this diagram to confirm your answers to 1(a).

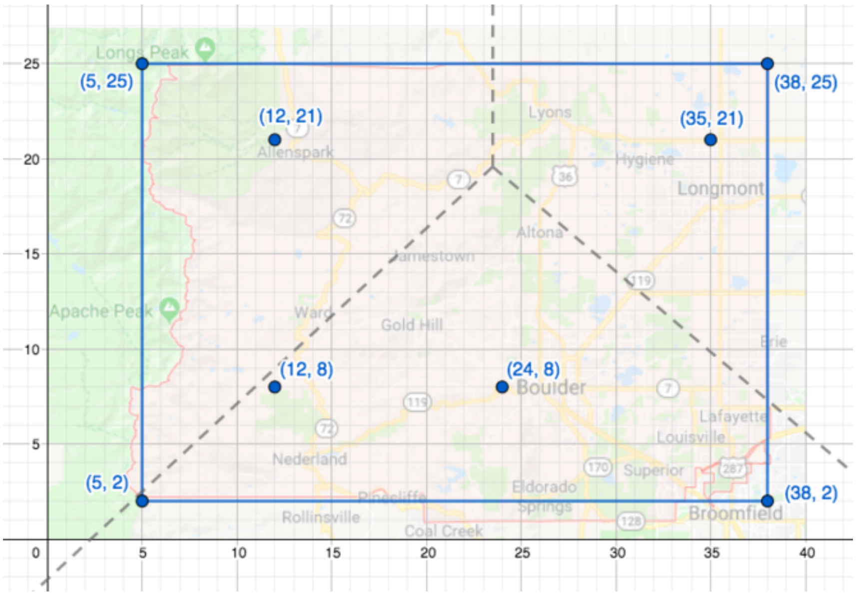
Step 2b: Find the equation of each edge using your knowledge of perpendicular bisectors.

Step 2c: Find the exact coordinates of the point that is equidistant from all three sites. Show your entire process.



Step 3: The meteorologists get additional funding to place a fourth precipitation collection site. Where should they place this site if they want the most accurate data possible for the whole county? Explain your reasoning.

Step 4: The diagram below shows the coordinates where the fourth site was placed, north of Nederland. In addition, Boulder County has been estimated as a rectangle with given coordinates. On your handout, use the incremental algorithm to construct the new Voronoi diagram that includes all four sites. Write the equations of any edges that are added.



The meteorologists want to estimate average precipitation across the entire county, because an annual precipitation below 46cm sharply increases the likelihood of wildfires.

Step 5a: Use your four-site diagram to calculate the area of each cell. Make sure to clearly show your entire process for determining any relevant coordinates and lengths.

Step 5b: Given the data below, calculate the average precipitation for all of Boulder County from June 2016 to June 2017. Each site's rainfall should be weighted by the area it represents.

Site	Boulder	Longmont	Allenspark	Nederland
Precipitation, June 2016-June 2017 (cm)	52.63	36.22	50.40	36.58

Step 5c: Should the meteorologists recommend that the wildfire danger level be increased? Explain.

Selected solutions

GeoGebra—Boulder County Rainfall—students can check intersection points and polygon areas.

Solution 2b: Equations of edges in three-site diagram:

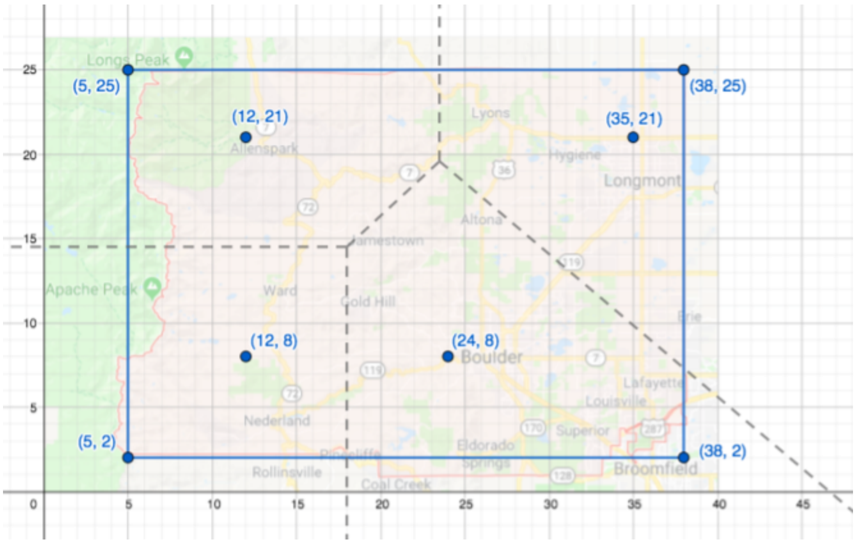
$$y - 14.5 = \frac{12}{13}(x - 18)$$

$$y - 14.5 = -\frac{11}{13}(x - 29.5)$$

$$x = 23.5$$

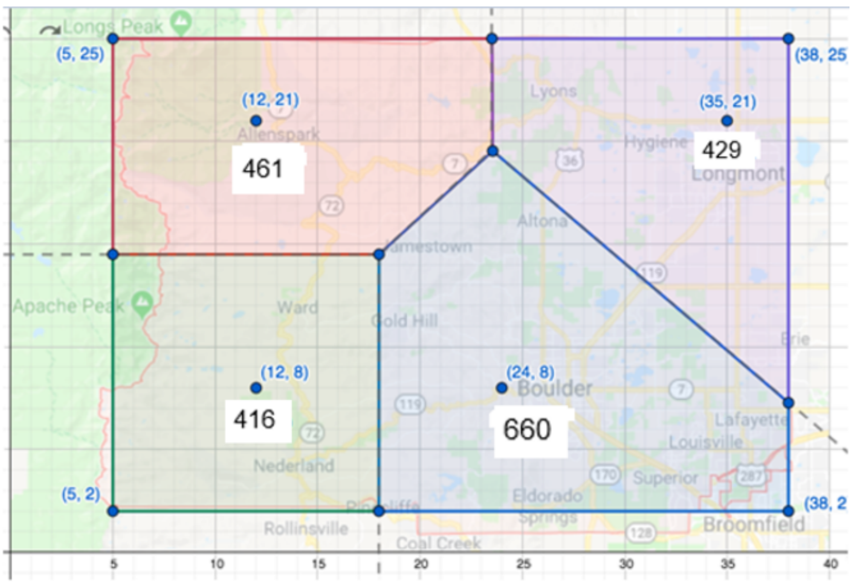
Solution 2c: Intersection point (solve system of equations or use technology): 23.5, 19.6.

Solution 4: Diagram with fourth site added:



New edges: $x = 18, \quad y = 14.5$

Solution 5a: Areas:



Site	Boulder	Longmont	Allenspark	Nederland
Precipitation, June 2016-June 2017 (cm)	52.63	36.22	50.40	36.58
Area (km^2)	660	429	461	416

$$Weighted \quad average = \frac{52.63 \times 660 + 36.22 \times 429 + 50.40 \times 461 + 36.58 \times 416}{660 + 429 + 461 + 416}$$

$$= 45.13 \quad cm/km^2$$

Differential equations, phase portraits and Euler's method (HL only)

The Mathematics: applications and interpretation HL course content covers exploring exact and approximate solutions to coupled systems of differential equations with an emphasis on long-term behaviour. It draws together several seemingly-disparate parts of the course and in doing so draws out a few of the many interconnections between different areas of mathematics.

In particular, the linked topics include: linear systems, eigenvalues and eigenvectors, complex numbers and exact and approximate methods for solving differential equations.

This section supports the teaching and learning of these interconnected areas.

Content-specific conceptual understandings from the guide

- Many physical phenomena can be modelled using differential equations and analytic and numeric methods can be used to calculate optimum quantities
- Phase portraits enable us to visualize the behaviour of dynamic systems

Differential equations, phase portraits and Euler's method

Preparing for the internal assessment exploration

How to use this section of the teacher support material

This teacher support material (TSM) is designed to support new and experienced teachers as they approach and implement the internal assessment (IA) with their students. It should be read in conjunction with the Diploma Programme *Mathematics: applications and interpretation guide* (published February 2019 for first examination in 2021), which contains the curriculum and assessment requirements at standard level (SL) and at higher level (HL).

This section offers suggestions and guidance for the implementation of the internally assessed component–the exploration. General regulation and procedures relating to internal assessment have not been reproduced here but can be found in the relevant section of the Diploma Programme *Assessment procedures*.

This publication contains material contributed by teachers to help other teachers and is intended to provide support and inspiration in a number of ways.

Another section of the TSM shows the application of the criteria in the assessment of explorations. It consists of explorations that have been assessed by experienced teachers using the assessment criteria. To look at the explorations teachers should go to the assessment section of this website and select "Assessed student work".

The internally assessed component in these courses is a mathematical exploration. This is a short report written by the student, based upon a topic chosen by him or her, and it should focus on the mathematics of that particular area. The emphasis is on mathematical communication (including formulae, diagrams, graphs and so on), with accompanying commentary, good mathematical writing and thoughtful reflection. A student should develop his or her own focus, with the teacher providing feedback via, for example, discussion and interview. This will allow all students to develop an area of interest for them, without a time constraint as in an examination, and will allow all to experience a feeling of success.

In addition to testing the assessment objectives of the courses, the exploration is intended to provide students with opportunities to increase their understanding of mathematical concepts and processes, and to develop a wider appreciation of mathematics. These are noted in the aims of the courses, in particular aims 6-11. It is intended that, by doing the exploration, students benefit from the mathematical activities undertaken and find them both stimulating and rewarding. It will enable students to acquire the attributes of the IB learner profile.

Teacher responsibilities

The teacher has nine main responsibilities.

During the process:

- to advise students in choosing an appropriate topic for an exploration
- to provide opportunities for students to learn the skills related to exploration work
- to ensure that students understand the assessment criteria and how they will be applied
- to encourage and support students throughout the research and writing of explorations
- to provide students with feedback on work completed at various stages of the exploration
- to give assistance to individual students in overcoming particular problems.

At the end of the process:

- to verify the accuracy of all calculations and to indicate on the exploration where mistakes have been made
- to assess the work accurately, annotating it appropriately to indicate where achievement levels have been awarded
- to ensure that students fully understand the strengths and weaknesses of the exploration.

It is important that relevant background information and comments regarding each criterion are included with the sample. It is recommended that this be indicated on the work itself.

Skills and strategies required by students

The exploration is a significant part of the course. It is useful to think of it as a developing piece of work, which requires particular skills and strategies. As a general rule, it is unrealistic to expect all students to have these specific skills and to follow particular strategies before commencing the course.

Many of the skills and strategies identified below can be integrated into the course of study by applying them to a variety of different situations both inside and outside the classroom. In this way, students can practise certain skills and learn to follow appropriate strategies in a more structured environment before moving on to working independently on their explorations.

Choosing a topic

It is essential that students choose a topic that can offer a productive route of inquiry, involve the use of relevant mathematics and engage the interest and enthusiasm of the student. The concept of the exploration should be introduced early in the course. Ideas for topics for explorations should be identified by students, in discussion with their teachers, as the course progresses.

For the majority of students, finding a suitable topic is the most difficult part of the process. Consequently, as soon as students are ready to begin work on their explorations, the teacher should allocate class time over two to three weeks to guide individual students through this process.

At the start of the process, teachers should discuss with students the overall form of the assessment as this may, in part, help to direct the flow of ideas and ultimately the focus of the exploration. Whole-class discussions where ideas are shared may help to lend focus to a topic. For students who have difficulty in choosing a topic, the following ideas may help.

- Identify an appropriate topic, taking in to consideration a student's own areas of interest
- Consider whether the focus will be analytic or an application of mathematics
- Look at the list of titles that have been submitted previously
- Look at the exemplars available here and consider the structure and features that have made them successful

Once a topic has been chosen:

- devise a focus that is well defined and appropriate
- make a detailed plan to give structure to the undertaking and the writing of the exploration
- ensure that the topic lends itself to a concise exploration
- if using data, ensure that enough data can be generated to ensure the mathematical techniques used are valid.

A list of previously submitted titles can be found in the appendices.

Presentation

- Express ideas clearly
- Identify a clear aim for the exploration
- Focus on the aim and avoiding irrelevance
- Structure ideas in a logical manner
- Include graphs, tables and diagrams at appropriate places

- Edit the exploration so that it is easy to follow
- Cite references where appropriate

Mathematical communication

- Use appropriate mathematical language and representation
- Define key terms and variables, where required
- Select appropriate mathematical tools (including information and communication technology)
- Set out any proofs in a logical way
- Express results to an appropriate degree of accuracy

Personal engagement

- Ask questions, make conjectures and investigate mathematical ideas
- Read about mathematics and research areas of interest
- Look for and create mathematical models for real-world situations
- Consider historical and global perspectives
- Explore unfamiliar mathematics

Reflection

- Discuss the implications of results
- Consider the significance of the exploration
- Look at possible limitations and/or extensions
- Make links to different fields and/or areas of mathematics
- Consider "what next?"

Use of mathematics

- Demonstrate knowledge and understanding
- Apply mathematics in different contexts
- Apply problem-solving techniques
- Recognize and explain patterns, where appropriate
- Generalize and justify conclusions

Use of technology

Assessment objective 4 for all DP mathematics courses is to "use technology accurately, appropriately and efficiently both to explore new ideas and to solve problems".

The exploration may offer opportunities for this objective to be achieved, although this is not a requirement for the exploration. In the exploration there are no limitations on the use of technology. It is reasonable, but not essential, to expect that students, when producing their explorations, will utilize technology in one or more ways.

Examples may include:

- any kind of calculators, hand-held or on the internet
- data-logging devices, simulations and modelling software
- word-processing packages, spreadsheets, graphics packages
- dynamic geometry software
- statistics packages or computer algebra packages.

Developing the exploration

Although the exploration is likely to be written in the second year of the course, students should be made familiar with the concept of the exploration at a very early stage. The specific planning and timing of the exploration will vary from school to school.

The following are suggestions that could be adopted at the different stages of the exploration.

Before students start the exploration

- Give out the criteria and stimuli early in the course and familiarize students with aims 6-11
- Give notice of a time frame for doing the exploration
- Encourage students to keep a record of ideas during the course (journal, notebook, blog)
- Encourage students to look for ideas everywhere (for example, reading mathematical material), and give access to such material (for example, TV, internet, other courses)
- Point out opportunities for exploring mathematics in everyday syllabus work
- Give students opportunities to practise mathematical writing
- Familiarize students with available technology

At the beginning of the exploration

- Look at examples from the TSM or other students' work
- Brainstorm and/or use mind-mapping activities to find a suitable topic
- Encourage the sharing and questioning of ideas
- Ensure that students have a clear, written focus before starting to write the exploration.

While students are doing the exploration

- Encourage self assessment and peer assessment
- Provide opportunities for discussion and questions between peers and with the teacher
- Provide appropriate feedback on the draft

After students have submitted the exploration

- Ensure that internal standardization between teachers takes place, including between SL and HL mathematics teachers
- Discuss with students the strengths and weaknesses of their exploration

Planning

1. Ensure that students have time to explore the mathematics.
2. Give a realistic deadline for submission of a draft of the written exploration.
3. Give a realistic deadline for feedback to the students.
4. Give a realistic deadline for final submission.

Developing a schedule

Deadlines for the completion of different stages of the exploration, preferably agreed to by both student and teacher, need to be firmly established. In particular, there need to be deadlines for the submission of:

- the exploration title and a brief description of the task, outlining the purpose of the exploration together with the strategies and techniques that will be used and, if applicable, how data is to be collected or generated, and how stimulus material has been used to generate ideas
- the draft of the exploration
- the finished exploration.

Long-term planning

The aim of long-term planning is to put the exploration into perspective in relation to the whole course.It should take into account:

- the sequencing of teaching units over the duration of the course
- those topics that are more applicable to the exploration
- appropriate places where the skills and strategies of the exploration can be introduced
- opportunities for students to record and develop ideas relevant to the exploration, for example, journals or blogs
- the resources available
- the role, if any, that the exploration will play in terms of a school's non-IB assessment(s)
- timetabling exploration deadlines into the school calendar.

Short-term planning

The aim of short-term planning is to provide a framework for the exploration so that students gain the maximum benefit from the experience.

It is expected that teachers will give help and guidance to the students while they are doing the exploration. Ten hours of class time should be allocated to management of the exploration work. Some of this time can be taken up with individual or group activities, where students learn some of the skills associated with exploration work. It is expected that students will spend additional time working on their explorations outside class time. Teachers should briefly discuss the exploration early during the course, so that students are aware of what is required and that this is an essential part of the course.

Stimuli

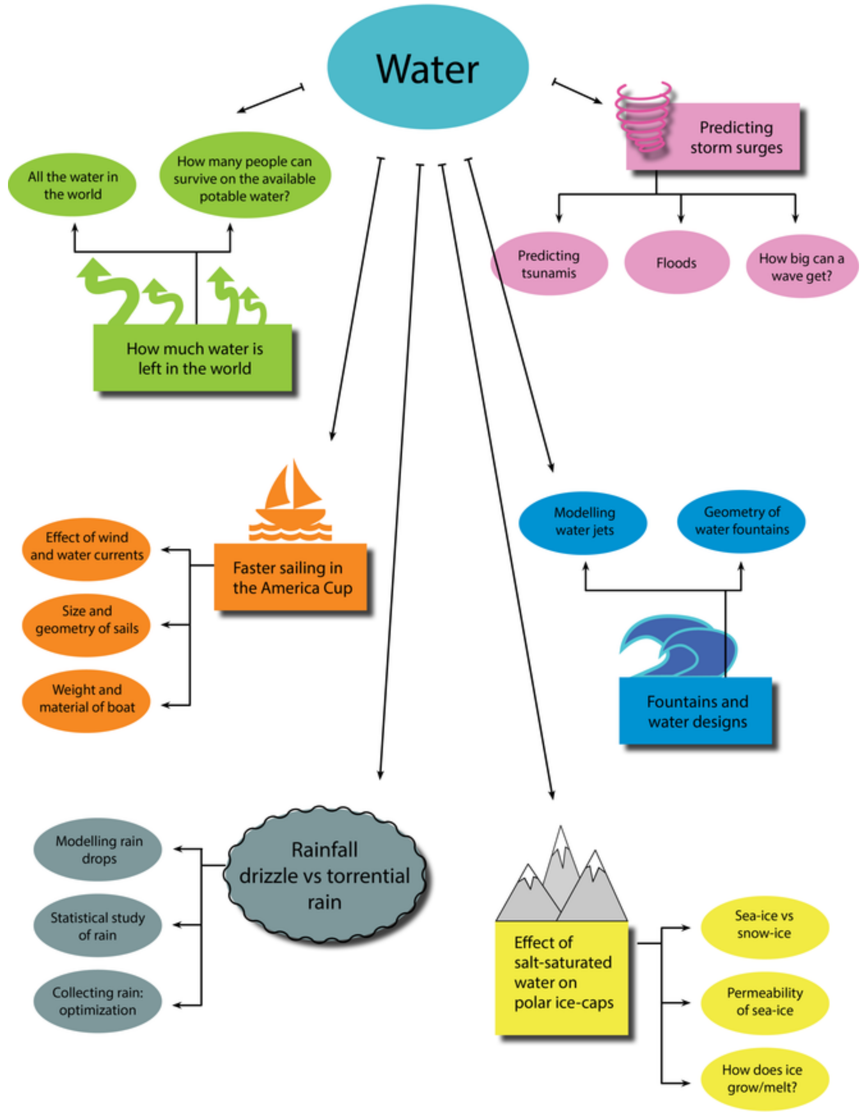
Students sometimes find it difficult to know where to start with a task as open-ended as this. While it is hoped that students will appreciate the richness of opportunities for mathematical exploration, it may sometimes be useful to provide a stimulus as a means of helping them to get started on their explorations.

Possible stimuli that could be given to the students

• sport	• architecture
• archaeology	• codes
• computers	• the internet
• algorithms	• communication
• cell phones	• tiling
• music	• population
• sine	• agriculture
• musical harmony	• viruses
• motion	• health
• e	• dance
• electricity	• play
• water	• pi (π)
• space	• geography
• orbits	• biology
• food	• business
• volcanoes	• economics
• diet	• physics
• Euler	• chemistry
• games	• information technology in a global society
• symmetry	• psychology

A possible mind map for the stimulus "water"

During introductory discussions about the exploration, the use of group discussion sessions can be useful to generate ideas. In particular, the use of a mind map has been shown to be useful in helping students to generate thoughts on this. The mind map below illustrates how, starting with the stimulus "water", some possible foci for a mathematical exploration could be generated.



Record keeping

Teachers are advised to keep detailed records about the exploration. It may be helpful to use forms such as forms A and B, for recording all the relevant information at the planning and feedback on the draft stages; these forms can be adapted for your own use. Please note that these are internal documents for the teacher and are not official IB forms.

Use of these forms is not mandatory and the forms can be adapted for individual circumstances. They have been suggested by a number of experienced teachers who have found them to be very useful. Form A (*Initial planning*) relates to the end of the initial planning stage and Form B (*Teacher feedback to student on draft*) relates to the teacher feedback following the students' submission of their draft explorations.

Form A—Exploration: Initial planning

Form B—Exploration: Teacher feedback to student on draft

Authenticity

Authenticity must be verified by signing the relevant form from the Diploma Programme *Assessment procedures* by both student and teacher.

By supervising students throughout, teachers should be monitoring the progress that individual students are making and be in a position to discuss with them the source of any new material that appears, or is referred to, in their explorations. Often students are not aware of when it is permissible to use material written by others or when to seek help from other sources. Consequently, open discussion in the early stages is a good way of avoiding these potential problems.

However, if teachers are unsure as to whether an exploration is the student's own work, they should employ a range of methods to check this fact. These may include:

- discussion with the student
- asking the student to explain the methods used and to summarize the results and conclusions
- asking the student to replicate part of the analysis using different data
- inviting the students to give a class presentation of their exploration.

Referencing and bibliography

Students should be made aware that direct or indirect use of the words of another person (in written, oral or electronic formats) must be acknowledged appropriately, as must any visual material used in the exploration that has been derived from another source. A student's failure to comply with this requirement will be viewed as plagiarism, and, as such, may be treated as a case of malpractice. Students should be familiar with the IB academic honesty policy, available on the programme resource centre.

The bibliography, or list of references, should include only those works (for example, books and journals) that the student has consulted while working on the exploration. An accepted form of quoting and documenting sources should be applied consistently. The major documentation systems are divided into two groups: parenthetical in-text name–date systems and numbered systems. Either may be used, provided this is done consistently and clearly.

Each work consulted, regardless of whether or not it has already been cited in the text as a reference, must be listed in the bibliography. The bibliography should specify: author(s), title, date and place of publication, and the name of the publisher, and should follow consistently one standard method of listing sources (for example, the Harvard system or the Vancouver system). Possible examples are:

Appadurai, A. 1990. "Disjuncture and difference in the global cultural economy". *Theory, Culture and Society.* Vol 7. Pp 295-310.

Miller, D. 2011. *Tales from Facebook*. Cambridge, UK. Polity Press.

Peterson, ADC. 2003. *Schools Across Frontiers: The Story of the International Baccalaureate and the United World Colleges. 2nd ed.* Chicago. Open Court Publishing Company.

Assessment criteria

Each exploration should be assessed against the following five criteria.

Criterion A	Presentation
Criterion B	Mathematical communication
Criterion C	Personal engagement
Criterion D	Reflection
Criterion E	Use of mathematics

The descriptions of the achievement levels for each of these five assessment criteria follow and it is important to note that each achievement level represents the **minimum** requirement for that level to be awarded. The final mark for each exploration is obtained by adding together the achievement levels awarded for each criterion A-E. It should be noted that the descriptors for criterion E are different for SL and HL.

The maximum possible mark is 20.

Applying the assessment criteria

The method of assessment used is criterion referenced, not norm referenced. That is, the method of assessing each exploration judges students by their performance in relation to identified assessment criteria and not in relation to the work of other students.

Each exploration submitted for mathematics is assessed against the five criteria A to E. For each assessment criterion, different levels of achievement are described that concentrate on positive achievement. The description of each achievement level represents the minimum requirement for that level to be achieved.

The aim is to find, for each criterion, the level descriptor that conveys most adequately the achievement level attained by the student.

Teachers should read the description of each achievement level, starting with level 0, until one is reached that describes a level of achievement that has **not** been reached. The level of achievement gained by the student is therefore the preceding one, and it is this that should be recorded.

For example, when considering successive achievement levels for a particular criterion, if the description for level 3 does not apply, then level 2 should be recorded.

For each criterion, whole numbers only may be recorded; fractions and decimals are not acceptable.

The highest achievement levels do not imply faultless performance, and teachers should not hesitate to use the extremes, including 0, if they are appropriate descriptions of the work being assessed.

A student who attains a high level of achievement in relation to one criterion will not necessarily attain high levels of achievement in relation to the other criteria. Similarly, a student who attains a low level of achievement for one criterion will not necessarily attain low achievement levels for the other criteria. Teachers should not assume that the overall assessment of the students will produce any particular distribution of marks.

It is expected that the assessment criteria will be available to students at all times. Descriptors of the achievement levels for each assessment criterion are given in the tables in the following section.

Students should be made aware that they will not receive a grade for mathematics if they have not submitted an exploration.

Achievement levels

Criterion A: Presentation

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	The exploration has some coherence or some organization.
2	The exploration has some coherence and shows some organization.
3	The exploration is coherent and well organized.
4	The exploration is coherent, well organized, concise.

The “presentation” criterion assesses the organization and coherence of the exploration.

A **coherent** exploration is logically developed, easy to follow and meets its aim. This refers to the overall structure or framework, including introduction, body, conclusion and how well the different parts link to each other.

A **well-organized** exploration includes an introduction, describes the aim of the exploration and has a conclusion. Relevant graphs, tables and diagrams should accompany the work in the appropriate place and not be attached as appendices to the document. Appendices should be used to include information on large data sets, additional graphs, diagrams and tables.

A **concise** exploration does not show irrelevant or unnecessary repetitive calculations, graphs or descriptions.

The use of technology is not required but encouraged where appropriate. However, the use of analytic approaches rather than technological ones does not necessarily mean lack of conciseness, and should not be penalized. This does not mean that repetitive calculations are condoned.

Criterion B: Mathematical communication

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	The exploration contains some relevant mathematical communication, which is partially appropriate.
2	The exploration contains some relevant appropriate mathematical communication.
3	The mathematical communication is relevant, appropriate and is mostly consistent.
4	The mathematical communication is relevant, appropriate and consistent throughout.

The “mathematical communication” criterion assesses to what extent the student has:

- used appropriate mathematical language (**notation, symbols, terminology**). Calculator and computer notation is acceptable only if it is software generated. Otherwise it is expected that students use appropriate mathematical notation in their work
- defined **key terms** and variables, where required
- used **multiple forms of mathematical representation**, such as formulae, diagrams, tables, charts, graphs and models, where appropriate
- used a **deductive method** and set out proofs logically where appropriate

Examples of level 1 can include graphs not being labelled, consistent use of computer notation with no other forms of correct mathematical communication.

Level 4 can be achieved by using only one form of mathematical representation as long as this is appropriate to the topic being explored. For level 4, any *minor* errors that do not impair clear communication should not be penalized.

Criterion C: Personal engagement

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	There is evidence of some personal engagement.
2	There is evidence of significant personal engagement.
3	There is evidence of outstanding personal engagement.

The "personal engagement" criterion assesses the extent to which the student engages with the topic by exploring the mathematics and making it their own. It is not a measure of effort.

Personal engagement may be recognized in different ways. These include thinking independently or creatively, presenting mathematical ideas in their own way, exploring the topic from different perspectives, making and testing predictions. Further (but not exhaustive) examples of personal engagement at different levels are given in the teacher support material (TSM).

There must be evidence of personal engagement demonstrated in the student's work. It is not sufficient that a teacher comments that a student was highly engaged.

Textbook style explorations or reproduction of readily available mathematics without the candidate's own perspective are unlikely to achieve the higher levels.

Significant: The student demonstrates authentic personal engagement in the exploration on a few occasions and it is evident that these drive the exploration forward and help the reader to better understand the writer's intentions.

Outstanding: The student demonstrates authentic personal engagement in the exploration in numerous instances and they are of a high quality. It is evident that these drive the exploration forward in a creative way. It leaves the impression that the student has developed, through their approach, a complete understanding of the context of the exploration topic and the reader better understands the writer's intentions.

Criterion D: Reflection

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	There is evidence of limited reflection.
2	There is evidence of meaningful reflection.
3	There is substantial evidence of critical reflection.

The "reflection" criterion assesses how the student reviews, analyses and evaluates the exploration. Although reflection may be seen in the conclusion to the exploration, it may also be found throughout the exploration.

Simply describing results represents **limited reflection**. Further consideration is required to achieve the higher levels.

Some ways of showing **meaningful reflection** are: linking to the aims of the exploration, commenting on what they have learned, considering some limitation or comparing different mathematical approaches.

Critical reflection is reflection that is crucial, deciding or deeply insightful. It will often develop the exploration by addressing the mathematical results and their impact on the student's understanding of the topic. Some ways of showing critical reflection are: considering what next, discussing implications of results, discussing strengths and weaknesses of approaches, and considering different perspectives.

Substantial evidence means that the critical reflection is present throughout the exploration. If it appears at the end of the exploration it must be of high quality and demonstrate how it developed the exploration in order to achieve a level 3.

Criterion E: Use of mathematics

The achievement levels and descriptors for criterion E are different for SL and HL.

SL only

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	Some relevant mathematics is used.
2	Some relevant mathematics is used. Limited understanding is demonstrated.
3	Relevant mathematics commensurate with the level of the course is used. Limited understanding is demonstrated.
4	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is partially correct. Some knowledge and understanding are demonstrated.
5	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is mostly correct. Good knowledge and understanding are demonstrated.
6	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Thorough knowledge and understanding are demonstrated.

The "Use of mathematics" SL criterion assesses to what extent students use mathematics that is **relevant** to the exploration.

Relevant refers to mathematics that supports the development of the exploration towards the completion of its aim. Overly complicated mathematics where simple mathematics would suffice is not relevant.

Students are expected to produce work that is **commensurate with the level** of the course, which means it should not be completely based on mathematics listed in the prior learning. The mathematics explored should either be part of the syllabus, or at a similar level.

A key word in the descriptor is **demonstrated**. The command term demonstrate means "to make clear by reasoning or evidence, illustrating with examples or practical application". Obtaining the correct answer is not sufficient to demonstrate understanding (even some understanding) in order to achieve level 2 or higher.

For knowledge and understanding to be **thorough** it must be demonstrated throughout.

The mathematics can be regarded as **correct** even if there are occasional minor errors as long as they do not detract from the flow of the mathematics or lead to an unreasonable outcome.

Students are encouraged to use technology to obtain results where appropriate, but **understanding must be demonstrated** in order for the student to achieve higher than level 1, for example merely substituting values into a formula does not necessarily demonstrate understanding of the results.

The mathematics only needs to be what is required to support the development of the exploration. This could be a few small elements of mathematics or even a single topic (or sub-topic) from the syllabus. It is better to do a few things well than a lot of things not so well. If the mathematics used is relevant to the topic being explored, commensurate with the level of the course and understood by the student, then it can achieve a high level in this criterion.

HL only

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	Some relevant mathematics is used. Limited understanding is demonstrated.
2	Some relevant mathematics is used. The mathematics explored is partially correct. Some knowledge and understanding are demonstrated.
3	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Some knowledge and understanding are demonstrated.
4	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Good knowledge and understanding are demonstrated.
5	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct and demonstrates sophistication or rigour. Thorough knowledge and understanding are demonstrated.
6	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is precise and demonstrates sophistication and rigour. Thorough knowledge and understanding are demonstrated.

The “Use of mathematics” HL criterion assesses to what extent students use **relevant** mathematics in the exploration.

Students are expected to produce work that is **commensurate with the level** of the course, which means it should not be completely based on mathematics listed in the prior learning. The mathematics explored should either be part of the syllabus, at a similar level or slightly beyond. However, mathematics of a level slightly beyond the syllabus is **not** required to achieve the highest levels.

A key word in the descriptor is **demonstrated**. The command term demonstrate means to make clear by reasoning or evidence, illustrating with examples or practical application. Obtaining the correct answer is not sufficient to demonstrate understanding (even some understanding) in order to achieve level 2 or higher.

For knowledge and understanding to be thorough it must be demonstrated throughout. Lines of reasoning must be shown to justify steps in the mathematical development of the exploration.

Relevant refers to mathematics that supports the development of the exploration towards the completion of its aim. Overly complicated mathematics where simple mathematics would suffice is not relevant.

The mathematics can be regarded as **correct** even if there are occasional minor errors as long as they do not detract from the flow of the mathematics or lead to an unreasonable outcome. **Precise** mathematics is error-free and uses an appropriate level of accuracy at all times.

Sophistication: To be considered as sophisticated the mathematics used should be commensurate with the HL syllabus or, if contained in the SL syllabus, the mathematics has been used in a complex way that is beyond what could reasonably be expected of an SL student. Sophistication in mathematics may include understanding and using challenging mathematical concepts, looking at a problem from different perspectives and seeing underlying structures to link different areas of mathematics.

Rigour involves clarity of logic and language when making mathematical arguments and calculations. Mathematical claims relevant to the development of the exploration must be justified or proven.

Students are encouraged to use technology to obtain results where appropriate, but **understanding must be demonstrated** in order for the student to achieve level 1 or higher, for example merely substituting values into a formula does not necessarily demonstrate understanding of the results.

The mathematics only needs to be what is required to support the development of the exploration. This could be a few small elements of mathematics or even a single topic (or sub-topic) from the syllabus. It is better to do a few things well than a lot of things not so well. If the mathematics used is relevant to the topic being explored, commensurate with the level of the course and understood by the student, then it can achieve a high level in this criterion.

Frequently asked questions about the IA

What is the difference between an exploration in mathematics and an extended essay in mathematics?

The criteria are completely different. It is intended that the exploration is to be a much less extensive piece of work than a mathematics extended essay. The intention is for students to "explore" an idea rather than have to do the formal research demanded in an extended essay.

How long should it be?

It is difficult to be prescriptive about mathematical writing. However, the *Mathematics: analysis and approaches* and the *Mathematics: applications and interpretation* guides state that 12 to 20 pages should be appropriate. An exploration may be less than 12 pages, however. A more common failing of mathematical writing is excessive repetition, and this should be avoided as such explorations will be penalized for lack of conciseness. It is recognized however that some explorations will require the use of several diagrams, which may extend them beyond the recommended page limit.

Are there any particular topics to be avoided?

A topic must be chosen so that the assessment criteria can be applied to it. Purely descriptive historical topics, for example, are not appropriate. See the appendices for a list of previously submitted titles.

When is a good time to introduce the exploration?

It is a good idea to mention it as early as possible, so that students are aware of the requirements, and to refer to it during the early part of the course. Certain topics may lend themselves more easily to exploration work, and teachers should try to make suggestions about this when appropriate. Ideally, the work on the exploration should start before the end of the first year.

Is any particular format for the exploration to be used?

No particular format is required. Students may write both the text of explorations and draw graphs and/or tables by hand, or explorations may be fully or partially word-processed. Either form is acceptable as long as the exploration is clearly legible. In recent years, students have used various forms of technology (for example, spreadsheets) to present data, construct tables and graphs, and perform calculations.

Does the exploration need a title?

It is good practice to have a title for all pieces of work. If the exploration is based on a stimulus, it is recommended that that the title not just be the stimulus. Rather, the title should give a better indication of where the stimulus has taken the student. For example, rather than have the title "Number patterns", the title could be "Number patterns—exploring patterns in final digits of prime numbers".

What should the target audience be for a student when writing the exploration?

The exploration should be accessible to fellow students.

Can the students use mathematics other than that which they have done in class?

Yes, but this must be clearly explained and referenced, and teacher comments should clarify this.

Can students use mathematics that is outside the syllabus?

It is not necessary to do this to obtain full or high marks. If students decide to explore mathematics outside the syllabus it is recommended that the level is commensurate with the syllabus.

Is interpretation of results a separate section or should comments be made during the exploration?

Commenting on and interpreting results at the point at which these are used enhances the communication and should be summarized in a conclusion. This may also apply to comments on the validity of results.

Must students use external resource material?

There is no requirement for the use of external resource material. However, students often find it necessary to obtain material from other sources (for example, for obtaining data, or for using formulae). In these cases, students should acknowledge these sources and list them in a bibliography and state any sampling processes used when using secondary data.

Can a student use data for an exploration that has already been used for other Diploma Programme internally assessed work (for example, the extended essay, fieldwork or experiments)?

This is to be discouraged, since it is unlikely that data collected for one particular use will lend themselves to being treated in a different manner. It may well be possible that students could use the data collected from work completed in other subjects, provided that it is analysed in a totally different manner. However, it is the student's responsibility to inform the teacher that these data have been collected for a different subject. The teacher must then ensure that no overlap occurs.

What is personal engagement?

The exploration is intended to be an opportunity for students to use mathematics to develop an area of interest to them rather than merely to solve a problem set by someone else. Criterion C (personal engagement) will be looking at how well the student is able to demonstrate that he or she has "made the exploration their own" and expressed ideas in an individual way.

What is the difference between precise and correct?

As outlined in criterion E (use of mathematics), "precise" mathematics requires absolute accuracy with appropriate use of notation. "Correct" mathematics may contain the occasional error as long as it does not seriously interfere with the flow of the work or give rise to conclusions or answers that are clearly wrong.

How can the teacher best monitor the work of students?

Having a schedule of due dates will help. It is also important that the teacher takes the time to review the work of students as the due dates come around. Developing a checklist of tasks and allowing for brief comment might help to keep open communication channels between students and the teacher.

Records of the progress made might best be kept by the students themselves, in the form of weekly journals. Teachers can simply read the journals and add a few brief comments. It may also be helpful to allow students to exchange journals for discussion or critique of work done during class time.

How much time should a student be spending on the exploration?

A total of 10 to 15 hours should be set aside for the exploration work in class. A portion of these hours can be spent on general class business (for example, reviewing policies and procedures, explaining the assessment criteria, reviewing progress, developing topics). Time spent on the exploration outside of class time should be in line with the normal homework expectation for 10 to 15 hours of class time.

What is the recommended target date for completion of the exploration?

This will, of course, vary from school to school depending upon several factors, not to mention other deadlines set within the Diploma Programme (for example, guided coursework, extended essays, laboratory reports). Teachers should also allow themselves plenty of time for the assessment process. The IB's deadline for samples of student work for moderation is in April for a May session school or October for a November session school. Therefore, it is not unreasonable for teachers to collect final explorations six to eight weeks prior to this deadline. Having an early deadline for completion may also make allowances for that student who undoubtedly will have a last-minute calamity.

Is there any way to deal with students who do little or no work on the exploration?

The obvious way to present to any student who is hesitant to make progress with their exploration is to emphasize the possible impact on the final assessment, with the exploration making up 20% of their final mark. If a student is reluctant to do any work at all, then perhaps a meeting of student, parents or guardians, the teacher and the Diploma Programme coordinator is advisable. At such a meeting, it would be appropriate to review the consequences of not submitting an exploration. Students should be made aware that they will not receive a grade for mathematics if they have not submitted an exploration.

It may also be helpful to develop a school or departmental policy for internal assessments, so that guidelines, due dates, expectations, consequences, and so on are made clear to both students and parents early in the course.

Some teachers are confused about how to apply the exploration assessment descriptors. Is guidance available?

In addition to this TSM, teachers can attend a mathematics workshop before it is time to assess the explorations of their students. Diploma Programme coordinators have information about workshops; such information can also be found on the IB public website (http://www.ibo.org). Another idea might be to ask advice from an experienced teacher. Obtaining a second opinion from an experienced teacher can be extremely helpful.

Can all students from one class submit explorations on exactly the same topic?

No. In fact, no two students should submit explorations that are exactly the same mathematically (they can, however, be from the same area or topic of mathematics, for instance "vectors"). The exploration is intended to be the sole work of an individual student. Whole class discussion can be used when generating ideas, selecting the topics for exploration, sharing research sources, acquiring the necessary knowledge, skills and understanding, and seeking peer-feedback on writing. However, the final exploration submitted must be the work of the individual student.

Can students in the same class/school use the same title for the exploration?

Yes, but the explorations must be different, based on the avenues followed by the student. As noted above, the title should give an idea of what the exploration is about.

Can SL and HL students use the same stimulus?

Yes, there is no reason to restrict any stimulus to a particular level, although the assessment of criterion E will be different.

Do teachers need to use stimuli?

No, but choosing a topic is often the most difficult part of the process for students, so it may be useful to provide stimuli as a means of helping students to get started on their exploration. Teachers are free to use their own stimulus material.

How many explorations should be done by a student during the course?

The exploration is a significant piece of work and, as such, the advice would be that there is no necessity to undertake more than one during the course. However, in line with the “Approaches to the teaching and learning” section of the two guides, students should be given many opportunities to use modelling and investigative techniques to develop the sorts of skills necessary to perform well in the exploration. The time allocated to the “toolkit” provides space to develop these skills.

Should the scope and sequence of the course be influenced by the exploration?

Ideally, it should not be. It is intended that the exploration should be a natural opportunity to develop ideas that students have become familiar with as a part of the course. However, if it is felt that particular skills are likely to be needed in order for students to undertake the exploration successfully, then a teacher or school may wish to consider this when deciding on the teaching sequence.

What constitutes a draft of the exploration and how much feedback can be given on it?

The draft is the only time prior to the student handing in the final exploration that the teacher can give formal feedback (written or otherwise) to the student. Teachers can, if they wish to, use or adapt Form B for this purpose. As the guide states, “teachers should read and give advice to students on one draft of the work. The teacher should provide oral or written advice on how the work could be improved, but not edit the draft”. This advice should be in terms of the way the work could be improved, but this first draft must not be heavily annotated or edited by the teacher. The next version handed to the teacher after the draft must be the final one. It is good practice for a teacher to provide informal feedback at all other stages of the exploration process.

How much help can the teacher give to the student with the mathematical content of the exploration?

If a student needs help with the revision of a particular topic because they are having some problems using this in their exploration, then it is permissible (indeed, this is good practice) for the teacher to give this help. However, this must be done in such a way that is not directly connected with the exploration.

If a school has a large number of students (or several classes) doing the exploration, must only one teacher mark all the explorations?

The exploration should be marked by the teacher who has supervised the class. However, teachers should be aware that moderation is applied to a school rather than to individual teachers. It is, therefore, of the utmost importance that teachers collaborate and agree on their marking standards. Guidance is available in the Diploma Programme *Assessment procedures*.

Should the student's final exploration be annotated?

As stated in the TSM, one of the teacher responsibilities is to assess the work accurately, annotating it appropriately to indicate where achievement levels have been awarded. It is essential that annotations are included on the student work to show why and where a level has been awarded. This includes assessing the mathematics and identifying and noting any errors. Without supporting comments, it is more difficult for the moderator to confirm the mark of the teacher.

Where can teachers receive more advice on the exploration?

Teachers should be aware that all questions on exploration work can be posted in the mathematics communities on the programme resource centre, and advice will then be offered by experienced teachers and the online faculty member. The programme resource centre also has many resources that have been posted by experienced teachers in the Communities, and these may provide a useful starting point for new teachers. However, it is important to understand that all opinions expressed by users of the programme resource centre are expressed strictly in their individual capacities, and not as representatives of the IB.

Preparing for papers 1 and 2

This section provides a useful resource in the form of a set of practice papers with examiner commentary (for use in the classroom, or for assessment purposes with students or for professional development by individual or groups of teachers), as well as some practical advice from examiners about papers 1 and 2. These practice papers with examiner commentary and the advice given below have been developed by experienced DP mathematics senior examiners and teachers.

The practice paper questions seek to indicate the ways in which new content could be assessed, the examiner commentary highlights some commonly occurring errors that students make and the accompanying markschemes with commentary seek to show clearly how and why the marks are allocated. Teachers should not regard these papers as additional specimen papers but rather a collection of questions on new topics in the form of an examination paper.

This resource can be used in a number of ways; some suggestions are:

- As the basis of professional development by individuals or groups of teachers
- As a learning and teaching resource to help students understand, for instance, what certain command terms imply about how students should express their answers
- As a learning and teaching resource to help students understand how marks are allocated and what examiners will be looking to see written down, particularly when using technology to find a solution
- As a whole paper to form additional practice for classwork or homework, or for summative or formative assessment
- As individual questions for discussion in the classroom to illustrate a teaching point or to highlight a common misconception.

Global notes

Teachers often have questions regarding *accuracy*, use of *units* and how students should *communicate* their answers in examinations, particularly when using technology to find solutions. The following global notes seek to clarify these areas which are exemplified within the practice papers by the examiner comments.

Accuracy

The rubric on the cover of examination papers states "Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures". Although final answers can be expressed to three significant figures, students should be encouraged to also write down the full answer prior to the rounding so that if their rounding is incorrect, marks may still be awarded for the correct full answer.

Students should not round intermediate answers as these approximations can combine to produce a final answer that is significantly inaccurate.

Follow-through marking means that full marks can be awarded in subsequent parts for using a candidate's rounded answer to an earlier part; however, students should be encouraged to use exact/unrounded values between question parts.

Units

Students should be encouraged to include units as appropriate in their answers, even when they are not explicitly requested in the question. Students should interpret the context and the units of any given data when finalising their answer. The omission of units will not always be penalized but some marks may be lost and hence students are advised to always include units where appropriate.

Communication

Students should be encouraged to present their responses in a coherent and structured format in order that the examiner can follow their solution and award appropriate marks. Although some method marks will be implied by correct answers, students should try to show all working. For instance this could include:

- defining new variables
- stating values/expressions they are entering into the calculator
- quoting intermediate values they find on their way to the correct answer.

In this way, appropriate marks can be awarded even if the final answer is incorrect. Stating "Using the GDC" in isolation will rarely be enough for an M1 mark, but a more specific statement clarifying the application being used (e.g. graphing; solver; statistical distributions etc.) and the values being used will clearly communicate the candidate's strategy to the examiner.

If students do not want a specific part of their response marked they should cross it out and the examiner will ignore it. However, students are advised to only do this if they intend to replace the response. Examiners frequently see incorrect or incomplete responses which have been crossed out and not replaced however the original M1 mark might have been awarded had the candidate not crossed it out.

Where a student has provided two responses to the same question, the examiner will only mark one. The student should indicate which response they wish to be marked (by crossing out or some other indication), but in lieu of this the examiner will mark the first response given.

Mathematics: applications and interpretation SL paper 1

Mathematics: applications and interpretation SL paper 1—annotated

Mathematics: applications and interpretation SL paper 1 markscheme

Mathematics: applications and interpretation SL paper 1 markscheme—annotated

Mathematics: applications and interpretation SL paper 2

Mathematics: applications and interpretation SL paper 2—annotated

Mathematics: applications and interpretation SL paper 2 markscheme

Mathematics: applications and interpretation SL paper 2 markscheme—annotated

Mathematics: applications and interpretation HL paper 1

Mathematics: applications and interpretation HL paper 1—annotated

Mathematics: applications and interpretation HL paper 1 markscheme

Mathematics: applications and interpretation HL paper 1 markscheme—annotated

Mathematics: applications and interpretation HL paper 2

Mathematics: applications and interpretation HL paper 2—annotated

Mathematics: applications and interpretation HL paper 2 markscheme

Mathematics: applications and interpretation HL paper 2 markscheme—annotated

Preparing for HL paper 3

This section gives some useful background information on the type and style of the questions from teachers involved in the development of the HL paper 3 and also advice from students involved in the trialling of the papers. This section should be read in conjunction with the external assessment details of the HL paper 3 in the subject guide.

General comments on the style of questions.

There will be two questions and the marks for each question will normally lie in the range 23 to 32.

The questions can be best thought of as extended closed problems.

The early parts of the question will be accessible to all students and there will be a general gradation of difficulty as the question progresses.

The length of the question will allow for a more exploratory type of question than is possible in papers 1 or 2.

The questions will usually include either unfamiliar mathematics or familiar mathematics used in unfamiliar settings. Hence, it is possible that the question may include topics outside of the guide. In these cases, sufficient support will be given in the question to ensure all parts are accessible.

Assessment objectives

The "Assessment objectives in practice" table within the course guide indicates the greater emphasis of inquiry approaches rather than knowledge and understanding compared with the other papers. This reflects that the questions asked might be focused on a small area of the syllabus with the ideas developed further than might be possible in a shorter question. There is also an emphasis on assessment objective 4 (Technology: use technology accurately, appropriately and efficiently both to explore new ideas and to solve problems) as for both subjects this paper requires the use of technology.

Approaches by course

The assessment objectives for problem solving and inquiry approaches state that students should be able to demonstrate the following.

- Problem-solving: recall, select and use their knowledge of mathematical skills, results and models in both abstract and real-world contexts to solve problems.
- Inquiry approaches: investigate unfamiliar situations, both abstract and from the real-world, involving organizing and analysing information, making conjectures, drawing conclusions, and testing their validity.

This distinction between abstract and real-world contexts is likely to be apparent when comparing the HL paper 3 from the two subjects, though there will always be an overlap between the two.

The Mathematics: analysis and approaches questions will frequently require the student to discover general patterns, to verify them and to informally justify or prove the result.

The Mathematics: applications and interpretation questions will frequently follow through the solution of a problem in a real-world context using mathematics developed in the course.

In some ways the Mathematics: analysis and approaches questions can be viewed as an extension of Criterion B: Investigating Patterns in the MYP, and the Mathematics: applications and interpretation questions an extension of Criterion D: Applying mathematics in real-life contexts.

Advice to students before taking the paper

It is easy for students to feel uncomfortable with extended questions, particularly if they have difficulties on the early parts. For this reason, the questions will be structured so that failure to do an early part of the question will not normally prevent a student from accessing later parts.

If a student encounters a difficulty while doing the questions they should look for points where they can "get back into" a question. Often the early parts will be "show that", which enable students to carry the result of one part into subsequent parts. Students should use the result given and not a different one that they might have found.

Though the questions will generally have an incline of difficulty, sometimes the nature of the question means a trickier part comes earlier and the final part might be a simple application of the result derived. The five minutes of reading time should be used to carefully read through the whole question.

The questions will always include a small introductory paragraph explaining their purpose–why they are exploring this particular problem. The questions do not have to completed in the order in which they are presented on the paper and an initial oversight of the problem may help candidates decide in which order to do the questions.

All of the skills needed for the AHL paper 3 are also needed for papers 1, 2 and the IA. However, some need to be more explicitly highlighted for the AHL paper 3. Some are general investigative approaches–although technology plays an important part in the AHL paper 3 so there are certain skills students should become familiar with in order to be successful.

Skills

- Solving systems of differential equations using Euler method
- Changing the parameters in a function, justifying accuracy of solutions by considering bounds
- Manipulation of matrices
- Choosing the appropriate statistical tests when given a dataset
- How to perform the statistical tests with the GDC
- Use of lists and list functions to manipulate datasets (for example, subtracting pairs of values to find a difference)
- Familiarity with the command terms which may require a written sentence, such as "suggest" and "comment"

Mathematics: applications and interpretation practice HL paper 3 questions

Mathematics: applications and interpretation practice HL paper 3 questions—annotated

A note on datasets in Mathematics: applications and interpretation

This paper may include analysis of a large dataset. At some point it is intended that the permitted technology will allow the candidates to be given the dataset. Currently though, they will need to enter the data themselves. The number of data points will be limited to what can reasonably be expected to be entered into a GDC in the time allocated, and the construction of the paper will include provision for this.

Teachers should make students aware that there will frequently be a value given (for example the mean of the data) which they can use to check they have entered the data correctly into their calculators.

List of previously submitted IA titles

The following list gives the titles of some explorations for the internal assessment that attained a variety of marks. Some titles are more descriptive than others and in most cases the original wording has been retained. These categories and titles are not an exhaustive list and have been chosen only as guidance.

Aesthetics

Calculating beauty–the golden ratio

Colour preferences

Daylight in a classroom–architectural design

Is my mirror showing an accurate image?

M.C. Escher: Symmetry and infinity of art

Modelling the surface area of the glass dome of the Galleria Vittoriio Emanuele II in Milan, Italy

Searching for the ideal sound

Shadows and height

Business and finance

A comparative study of shares, real estate, bonds and banks

Analysis of stock market changes

Applications of calculus to the economics of firms

Buying a car or a house–payment options

Code breaking

Economic development and levels of income

Finding the lowest values of the dimensions of differently shaped storage rooms using differential calculus and optimisation

International phone call pricing

Statistics on flight information for an international airline

Food and drink

Costs of products bought online compared to local grocery stores

Dine in or dine out?

How many peas are there in a 500 gram box of peas?

Jelly bean study

The cookie problem–taste is all-important

The operation of a tuck shop

The volume of an egg

What is the greatest candy bar in the world?

Health and fitness

A comparison between calorie intake and gender

A comparison between lung capacity, age, weight and body fat

Aids awareness in Maseru

Blood pressure

Breakfast and school grades

Breast and cervical cancer–ethnic comparison

Infant mortality

Investigating reaction times

The SIR model in relation to world epidemics

Geometry and trigonometry

Geodesic domes

Graph theory–finding the shortest path

Newton-Raphson

Origami applications to mathematics

Sine waves in pitch frequencies

Spanning trees

Spherical geometry

Stacking bricks

The ideal cut of a diamond

The Ferris wheel

The open Knight’s Tour on a chessboard

Topography and distance

Nature and natural resources

Airfoil and lift force

Analysis of the cost and utility of gas versus electricity in an average domestic situation

Animal population

Calculating the time of sunrise and sunset

Chaos theory: universal prediction

Counting weeds

Earthquakes–can they be predicted?

Florence Nightingale and modelling spread of disease

Graphing the Pharmacokinetic Profile

How does population density affect the transmission of Ebola?

Is the swell of the sea influenced by the temperature?

Modelling Arctic Sea ice cover

Modelling rainfall

Modelling the cooling of a cup of tea

Optimum dimensions of an aluminium drink can

Predicting cooling times

Rainfall compared to grape vine yield

Statistical investigation of leaves

The quality of local water

The SIR model in relation to world epidemics

The volume of an egg

Sunspot cycles

What is the relationship between the duration of drainage and water height in my bathtub?

Number

Approximation of pi

Cyclic situations and patterns through happy numbers

e, π and φ: are they related?

The golden number phi

What is e?

Euler's totient theorem

People

Assuming a person has an 85% chance of meeting a soul mate during their lifetime, what does that mean about the number of potential soul mates in the world?

Correlation between divorce rate and financial uncertainty

Does gender influence choice of favourite animal?

Does the electoral college in the US truly represent the political choice of the people?

Effect on tipping percentages

Exploring the gamblers' fallacy–why it can cause fatal decisions

Is film genre choice more dependent on nationality or gender?

Gender-based discrimination

Left-handed students

Memory

Perception of time

Relationship between a country's human development index and infant mortality rate

Relationship between GDP and fertility rate in countries across the world

Relationship between income inequality and rate of corruption in a country

Relations between international and bilingual students: jobs, pocket money and spending behaviour

Relationship between unemployment and criminality in Sweden from 1988-1999

Relationship between women's secondary education and fertility rates in developing countries

Statistical comparison of the number of words in a sentence in different languages

The birthday paradox

When can I use "swimmed" and "knowed" correctly?

Voter turnout

Sport and leisure

Baseball bat speed compared with body weight

Body proportions for track and field events

Does the team win when it was the dominating team during the match?

Effective short corners in hockey

Exploring card counting in blackjack using probability

Factors affecting athletic performance

Has sports performance improved more on land or in water?

Height, weight and swimming performance

How does the amplitude of a ski turn affect the speed of the skier?

How far do tennis balls roll?

The geometry involved in billiards

Modelling musical chords

Modelling the jump of a horse

Practice makes perfect

Relationship between skiing ability and distance travelled to ski

Resistance of fishing line

Rollerblading and the maths behind it

The Monty Hall problem

The Tower of Hanoi puzzle

Video games and response times

Will female swimmers ever overtake male swimmers?

Travel and transport

Cost efficiency of vehicles

Driving skills

How many bicycles are there in Amsterdam?

Petrol prices

Public transportation costs and car usage: a personal comparison

Running late and driving habits

Seat belt use

The effect of blood alcohol content law on the number of traffic collisions in Sacramento

Traffic study of Schiphol International Airport

Transport safety in town centres

Glossary of terminology: Graph theory

Introduction

Teachers and students should be aware that many different terminologies exist in graph theory, and that different reference sources may employ different combinations of these. Examples of these include:

- vertex/node/point
- edge/link/line
- degree of a vertex/order of a vertex
- multiple edges/parallel edges
- loop/self-loop.

In IB examination questions, the terminology used will be as presented in the guide. For clarity, these terms are defined below.

Term	Definition
Adjacency matrix	A square matrix whose entries indicate whether pairs of vertices are adjacent or not in the graph. The $(i, j)^{\text{th}}$ entry of A^k gives the number of walks from i to j that traverse exactly k edges.
Adjacent edges	Two edges that share a common vertex.
Adjacent vertices	Two vertices joined by an edge.
Circuit	A walk that begins and ends at the same vertex and has no repeated edges.
Complete graph	A simple graph in which each pair of vertices is joined by an edge.
Connected graph	A graph in which each pair of vertices is joined by a path.
Cycle	A walk that begins and ends at the same vertex and has no other repeated vertices.
Degree of a vertex	The number of edges joined to the vertex.
Directed graph	A graph whose edges have an indicated direction.
Eulerian circuit	A circuit that contains every edge of a graph.
Eulerian trail	A trail that contains every edge of a graph.
Graph	Consists of a set of vertices and a set of edges.
Hamiltonian cycle	A cycle that contains all the vertices of a graph.
Hamiltonian path	A path that contains all the vertices of a graph.
In degree and out degree of a vertex	For a vertex of a directed graph, "in degree" refers to the number of edges leading **to** the vertex, and "out degree" refers to the number of edges leading **from** the vertex.
Loop	An edge joining a vertex to itself.
Minimum spanning tree	A spanning tree of a weighted graph that has the minimum total weight.
Path	A walk with no repeated vertices.

Term	Definition
Simple graph	An undirected graph without loops, and one edge at most, between any pair of vertices.
Spanning tree of a graph	A subgraph that is a tree, containing every vertex of the graph.
Strongly connected graph	A directed graph in which every vertex can be reached from every other vertex.
Subgraph	A graph within a graph.
Trail	A walk in which no edge appears more than once.
Transition matrix	A matrix whose $(i, j)^{\text{th}}$ entry gives the probability that an element moves from the j^{th} state to the i^{th} state in a single step of the process.
Tree	A connected graph that contains no cycles.
Undirected graph	A graph whose edges are bidirectional.
Walk	A sequence of linked edges.
Weighted adjacency table	A table in which the $(i, j)^{\text{th}}$ entry gives the weight of the edge connecting vertex i and vertex j in the corresponding graph.
Weighted graph	A graph in which each edge is allocated a number or weight.

Further reading

For teachers and students

The following is a list of suggested reading for students and for teachers. The suggestions for students can be useful starting points for the internal assessment and are interesting recreational reading. The lists are not an exhaustive and are not recommended textbooks.

Abbott, E. 1992. *Flatland: A Romance of Many Dimensions.* New York, USA. Dover Publications Inc.

Acheson, D. 2010. *1089 and All That: A Journey into Mathematics.* Oxford, UK. Oxford University Press.

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Eastaway, R and Wyndham, J. 2005. *Why do Buses come in Threes?: The Hidden Mathematics of Everyday Life.* London, UK. Portico.

Gardner, M. 2008. *Hexaflexagons, probability paradoxes, and the Tower of Hanoi: Martin Gardner's first book of mathematical puzzles and games.* Washington, DC, USA. Mathematical Association of America.

Grinstead, C and Snell, J. 2009. *Ginstead and Snell's Introduction to Probability.* Gainesville, USA. University Press of Florida.

Hemmings, R and Tahta, D. 1992., *Images of Infinity.* St Albans, UK. Tarquin Publications.

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Korner, T. 1996. *The Pleasures of Counting.* Cambridge, UK. Cambridge University Press.

Munroe, R. 2015. *What if? Serious Scientific Answers to Absurd Hypothetical Questions.* London, UK. John Murray.

Stewart, I. 2006. *How to Cut a Cake: and other mathematical conundrums.* Oxford, UK. Oxford University Press.

Stewart, I. 2013. *17 Equations that Changed the World.* London, UK. Profile Books.

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Project zero. *Visible Thinking*. http://www.visiblethinkingpz.org/VisibleThinking_html_files/VisibleThinking1.html. Accessed 26.04.2018.

For teachers

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Burge, B, Lenkeit, J and Sizmur, J. 2015. PISA in Practice–Cognitive activation in *Maths: How to use it in the classroom.* NFER. Available at: https://www.nfer.ac.uk/publications/PQUK04/ (accessed 28 March 2018)

De Lange, J. 2003. "Mathematics for literacy" in *Madison B (Ed), Quantitative Literacy: Why Numeracy Matters for Schools and Colleges.* Available at: https://www.maa.org/sites/default/files/pdf/QL/pgs75_89.pdf (accessed 28 March 2018)

Eichler, A and Zapata-Cardona, L. 2016. *Empirical Research in Statistics Education (ICME-13 Topical Surveys).* New York, USA. Springer.

Ellenberg, J. 2015.. *How Not to be Wrong: The Power of Mathematical Thinking*. New York, USA. Penguin Random House.

Evan, S. 2011. *Graph Algorithms (2nd edition).* Cambridge, UK. Cambridge University Press.

Greefrath, G and Vorholter, K. 2016. *Teaching and Learning Mathematical Modelling: Approaches and Developments from German Speaking Countries (ICME-13 Topical Surveys).* New York, USA. Springer.

Hegedus, S, Laborde, C, Brady, C, Dalton, S, Siller, S, Tabach, M, Trgalova, J and Moreno-Armella, L. 2016. *Uses of Technology in Upper Secondary Mathematics Education (ICME-13 Topical Surveys).* New York, USA. Springer.

Ritchhart, R, Church, M and Morrison, K. 2011. *Making Thinking Visible*. San Francisco, USA. Jossey Bass.

Rosa, M, D'Ambrosio, U, Clark Orey, D, Shirley, L, Alangui W, Palhares, P and Gavarrete, M. 2016. *Current and Future Perspectives of Ethnomathematics as a Program (ICME-13).* New York, USA. Springer.

Rouncefield, M and Holmes, P. 1989. *Practical Statistics.* London, UK. Palgrave Macmillan.

Trudeau, R. 2003. *Introduction to Graph Theory.* New York, USA. Dover Publications.

Wathall, J. 2016.. *Concept-based Mathematics: Teaching for deep understanding in secondary classrooms.* Thousand Oaks, USA. Corwin.

Wiliam, D and Leahy S. 2015. *Embedding formative assessment: Practical Techniques for K-12 classrooms.* La Vergne, USA. LSI.

Willingham, DT. 2007. *Critical Thinking: Why Is It So Hard to Teach?* American Educator, 31, 8-19.

Voronoi diagrams

Consortium for mathematics and its applications, *Mathematics: modelling our world* www.comap.com/mmow/Course2.html. [Accessed on 18 June 2018].

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Drysdale, S. 1993. *Voronoi Diagrams: Applications from Archaology to Zoology*. https://www.ics.uci.edu/~eppstein/gina/scot.drysdale.html. [Accessed on 18 June 2018].

Lynch P (2017). *How Voronoi diagrams help us understand the world.* The Irish Times, 23 January. https://www.irishtimes.com/news/science/how-voronoi-diagrams-help-us-understand-our-world-1.2947681. [Accessed on 18 June 2018].

Videos

Video 1

Preparing for the HL paper 3

Video 2

Technology in the DP mathematics classroom

Video 3

The effective use of the mathematics toolkit

Video 4

Course selection: communicating choices

Updates to the publication

This section outlines the updates made to this publication over the past two years. The changes are ordered from the most recent to the oldest updates. Minor spelling and typographical corrections are not listed.

Corrections for February 2023

The toolkit > Using technology

Amendment in response to stakeholder feedback.

"Financial applications of geometric sequences and series"

In this unit planner, the solution for question 3 (a) was changed.

There was an error in the original solution where the quarterly payment period had been calculated twice. The solution has been changed to illustrate the approach that can be taken to calculate real interest earned when the nominal interest rate period and the compounding periods are not the same.

"Amortization and annuities"

In the second sentence for question 4 in this unit planner, the term "monthly" was changed to "annually" so the sentence now reads: "At the end of every year Meredith will deposit money into an annuity fund, earning 6% per annum compounding annually".